Dot Product:

$\vec{u} = \langle u_1, u_2 \rangle$

$\vec{v} = \langle v_1, v_2 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$ (vector) • (vector) = number

<u>RE.</u> show that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

<u>Properties</u>:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{0} \cdot \vec{v} = 0$
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
5. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$ $\quad c \in \mathbb{R}$.

You can verify all of these.

<u>Ex.</u> $\langle 4, 5 \rangle \cdot \langle 2, 3 \rangle$

$\langle 2, -1 \rangle \cdot \langle 1, 2 \rangle$

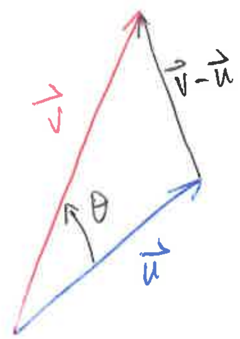
$\langle 0, 3 \rangle \cdot \langle 4, -2 \rangle$

<u>Ex.</u> $\vec{u} = \langle -1, 3 \rangle$, $\vec{v} = \langle 2, -4 \rangle$, $\vec{w} = \langle 1, -2 \rangle$

a.) $(\vec{u} \cdot \vec{v})\vec{w}$

b.) $\vec{u} \cdot 2\vec{v}$

c.) $\|\vec{u}\|$



The angle θ between two nonzero vectors $\vec{u}$ and $\vec{v}$ is always between 0 and π.

$$0 \le \theta \le \pi.$$

$$\boxed{\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}}$$

Proof. Consider the triangle above.

Law of cosines says

$$\|\vec{v}-\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\|\vec{v}-\vec{u}\|^2 = (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u}) = \|\vec{v}\|^2 - 2\vec{v}\cdot\vec{u} + \|\vec{u}\|^2, \text{ so}$$

$$\cancel{-2}\,\vec{v}\cdot\vec{u} = \cancel{-2}\,\|\vec{u}\|\|\vec{v}\| \cos\theta \quad \leftarrow$$

we also get this very useful formula!

or

$$\cos\theta = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|\|\vec{v}\|} \quad \square \quad \ddot\smile$$

Ex. $\vec{u} = \langle 4,3\rangle$, $\vec{v} = \langle 3,5\rangle$ Find θ.

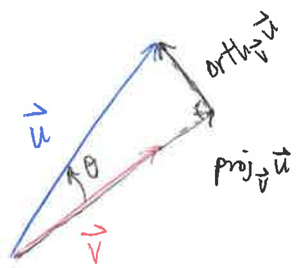
$$\cos\theta = \frac{27}{5\sqrt{34}} \Rightarrow \theta \simeq 0.3869 \text{ rad}$$

Def'n. Two angles are orthogonal if the angle θ between them is $\pi/2$. i.e., they are perpendicular. write $\vec{u}\perp\vec{v}$.

Consequence. $\vec{u}\perp\vec{v}$ if and only if $\vec{u}\cdot\vec{v} = 0$.

Ex. $\langle 2,-3\rangle$ and $\langle 6,4\rangle$

Projections and Stuff.



Given vectors $\vec{u}$ and $\vec{v}$, $\vec{u}$ can be decomposed into components that closely reflect the direction of $\vec{v}$.

$$\vec{u} = \vec{w}_1 + \vec{w}_2$$

$$\vec{w}_1 = \text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\,\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

Idea: $\dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector in the direction of $\vec{v}$.

$\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$ is the amount needed to shrink or stretch $\vec{w}_1$ to make the projection orthogonal.

$$\vec{w}_2 = \vec{u} - \vec{w}_1 = \vec{u} - \text{proj}_{\vec{v}}\vec{u} =: \text{orth}_{\vec{v}}\vec{u}.$$

~~Exam~~ Ex. $\vec{u} = \langle 3,-5\rangle$, $\vec{v} = \langle 6,2\rangle$

Find $\text{proj}_{\vec{v}}\vec{u}$, then decompose $\vec{u}$ into a sum $\vec{w}_1 + \vec{w}_2$ as above.

Application problems.

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

$$\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} = \|\vec{u}\|\cos\theta, \quad \text{therefore } \|\text{proj}_{\vec{v}}\vec{u}\| = \left|\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right|.$$