Math 123: Trigonometry

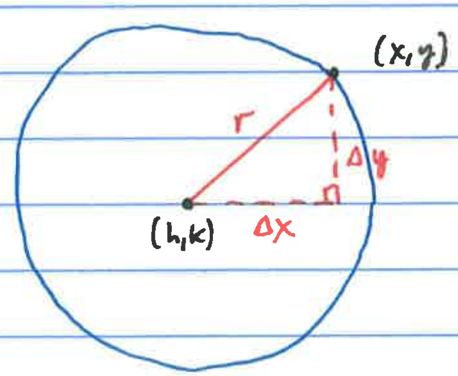
Etimologically, trigonometry means "the measure of triangles" as derived from Greek. We'll spend much of this semester studying triangles, but we begin our course by studying circles.

Review: The Unit Circle

Recall from algebra that a circle in the xy-plane, centered at the point (h,k) with radius r has the equation

$$(x-h)^2 + (y-k)^2 = r^2$$

This is just the Pythagorean Theorem:



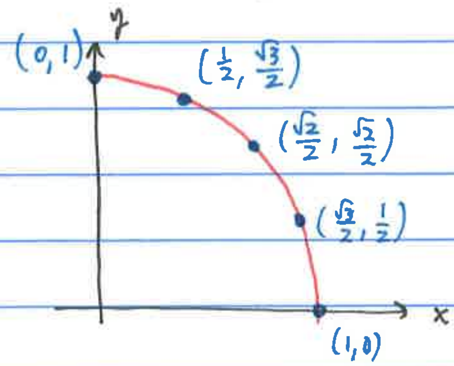
$$(\Delta x)^2 + (\Delta y)^2 = r^2$$
$$(x-h)^2 + (y-k)^2 = r^2$$

If $(h,k) = (0,0)$ and $r=1$, then the equation becomes

$$\boxed{x^2 + y^2 = 1}$$

This special circle is called the Unit Circle.

Let's examine the first quadrant of the unit circle:



There are five special points:

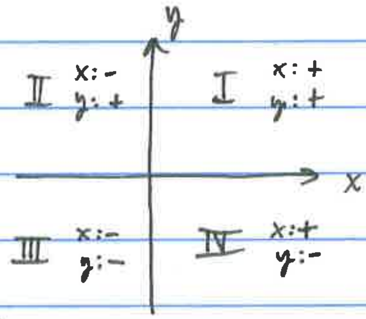
x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
y	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Ex. $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$ ✓

$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = \frac{4}{4} = 1$ ✓

$0^2 + 1^2 = 0 + 1 = 1$ ✓

Recall that the x- and y- axes separate the plane into 4 quadrants:



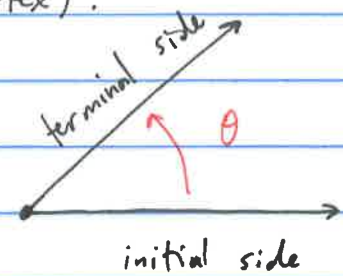
We can use reflections to fill in all of the "special points" of the unit circle.

Ex. You should do it!

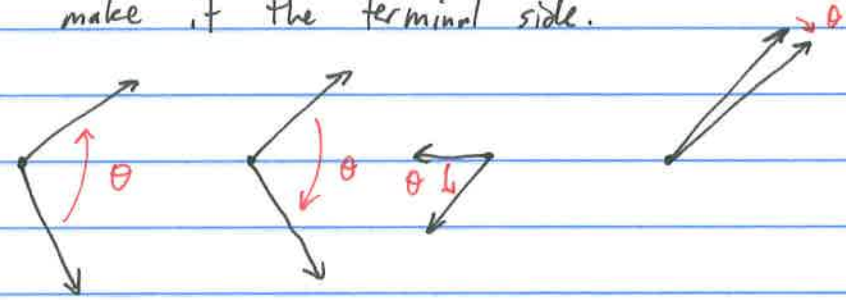
Chapter 1: Trigonometry

1.1: Angles: radian and degree measure.

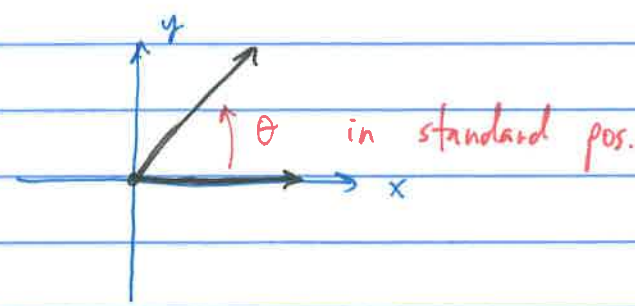
An angle is made of two rays that share their initial point (the vertex).



The angle θ tells you how far you would rotate the initial side to make it the terminal side.



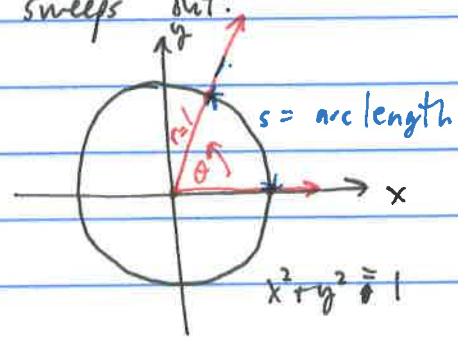
If we introduce an xy-plane (x- and y-axes), then θ is in standard position if its initial side corresponds with the positive x-axis:



An angle is __positive__ if it sweeps out the angle in the counter-clockwise direction.
An angle is __negative__ if it sweeps clockwise.

Radian Measure.

We need a way to measure angles. The most obvious way is to consider how much of a circle (the unit circle!) an angle sweeps out.



__Def'n.__ One __radian__ is the measure of a central angle θ that intercepts an arc s equal to the radius r of the circle:

$$\theta = \frac{s}{r}$$

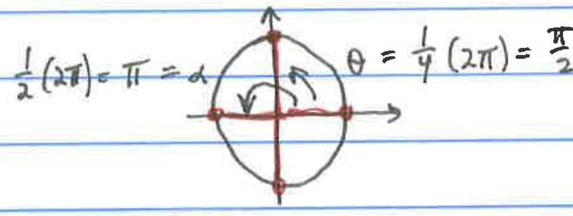
* Both s and r are measured in length, so a radian has __no__ __units__!

If $r=1$, then $\theta = s$.
If $r=s$, then $\theta = 1$.

Since the circumference of the unit circle is 2π, there are 2π radians in a complete revolution.

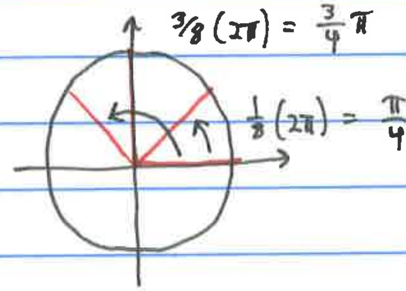
Since we know there are 2π radians in a circle, we can break the circle (disc) into equivalent pieces to find some special angles.

<u>Ex.</u> Break the unit circle into 4 pieces:

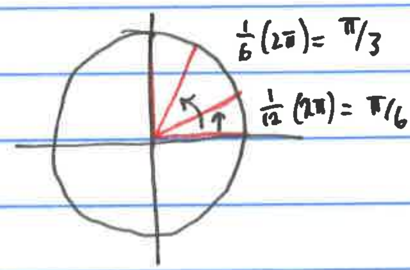


etc.

8 pieces:



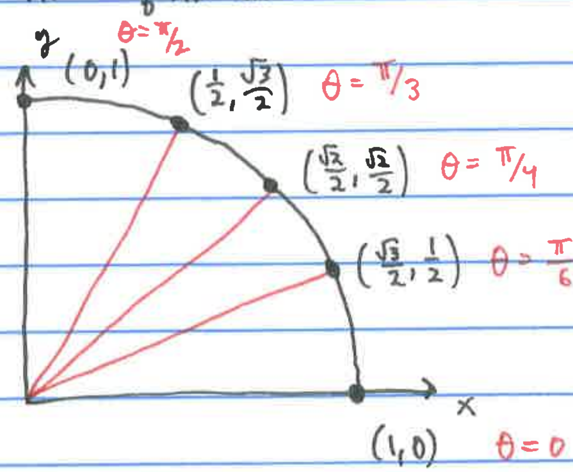
12 pieces:



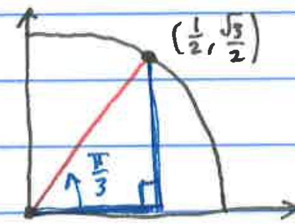
etc.

Now look at the first quadrant of the unit circle again:

We can now associate our "special points" on the unit circle to "special angles" in radians



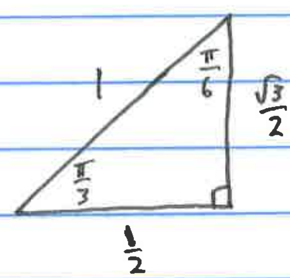
Notice: If we pick any 1 point we can draw a right triangle.



This triangle has one angle of $\frac{\pi}{2}$ (the "right" one) and one angle of $\pi/3$. Since the angles in a triangle always add up to π, the third angle must be

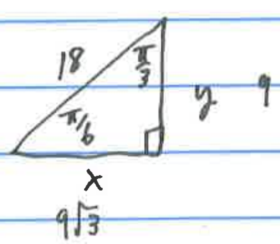
$$\pi - \frac{\pi}{2} - \frac{\pi}{3} = \frac{6\pi}{6} - \frac{3\pi}{6} - \frac{2\pi}{6} = \boxed{\frac{\pi}{6}}$$

This gives us the "Special triangle":



Notice that the longest side is accross from the largest angle, and similarly for the smallest.

Every $\frac{\pi}{6}$-$\frac{\pi}{3}$-$\frac{\pi}{2}$ (30-60-90) right triangle has sides with these same ratios.

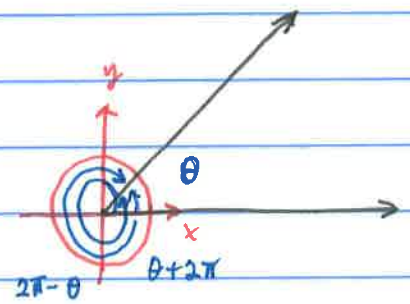
Ex. 

Find the lengths of the legs.

$$\frac{18}{x} = \frac{1}{\sqrt{3}/2} \Rightarrow x = 9\sqrt{3}$$

and $\frac{18}{y} = \frac{1}{1/2} \Rightarrow y = 9$

Let's go back and think about angles again.



All of these angles are coterminal.

We can always find another coterminal angle by adding or subtracting 2π, or ~~multiplying~~ multiples of 2π.

Ex. Consider the angle $\frac{37}{2}\pi$. Find a coterminal angle between 0 and 2π.

$$\frac{37}{2}\pi = \frac{36}{2}\pi + \frac{\pi}{2} = 18\pi + \frac{\pi}{2}$$

$18\pi = 9(2\pi)$, or 9 full revolutions. Therefore $\frac{\pi}{2}$ is coterminal to $\frac{37}{2}\pi$, and lies in the interval $[0, 2\pi)$.

* This example shows that we can always reduce our angles to ones that live within a single revolution of the circle.

We will use this property extensively in our study of trig. functions.

Ex. Find a positive and negative coterminal angle.

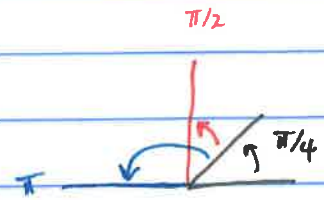
$\theta = \frac{5\pi}{6}$

$\theta_1 = \frac{5\pi}{6} + 2\pi = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}$

$\theta_2 = \frac{5\pi}{6} - 2\pi = \frac{5\pi}{6} - \frac{12\pi}{6} = -\frac{7\pi}{6}$

Two positive angles that add up to $\frac{\pi}{2}$ are called <u>complementary</u>.
Two positive angles that add up to π are called <u>supplementary</u>.

<u>Ex</u>. $\theta = \frac{\pi}{4}$ Find its complement and supplement.



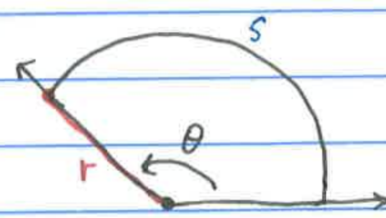
$$\left.\begin{aligned}\theta_c &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{2\pi}{4} - \frac{\pi}{4} = \pi/4 \\ \theta_s &= \pi - \frac{\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4} = 3\pi/4\end{aligned}\right\}$$

<u>Ex</u>. $\theta = \frac{5\pi}{6}$ $\frac{6\pi}{6} > \pi/2$, so it has no complement.

$$\theta_s = \pi - \frac{5\pi}{6} = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}.$$

<u>Arc Length</u>

Consider a circle w/ radius r and central angle θ.
Can we find the arc length of the segment of the circle subtended by the angle?



$\theta = \frac{s}{r} \Rightarrow \boxed{s = r\theta}$
Arc length formula.

<u>Ex</u>. $\theta = 1.5$, $r = 4$
Then $s = 4(1.5) = \boxed{6.}$

Linear and Angular Speeds

Consider a particle moving around a circular arc of radius r. If s is the arc length traveled in time t, then the linear speed v of the particle is:

$$\boxed{v = \frac{s}{t}}, \text{ or arc length per unit time.}$$

If θ is the angle subtended by the particle, then the angular speed ω of the particle is:

$$\boxed{\omega = \frac{\theta}{t}}, \text{ or radians per unit time.}$$

Ex. ~~The one~~ The second hand on a clock is 10 cm long. Find the linear speed of its tip as it travels around the clock. in cm/s.

One revolution is: $s = r\theta = 10\text{ cm}(2\pi) = 20\pi$ cm

One revolution takes 1 min = 60 sec

Therefore the ~~ang~~ linear speed is $v = \frac{20\pi \text{ cm}}{60 \text{ sec}} = \boxed{\frac{\pi}{3} \text{ cm/sec.}}$

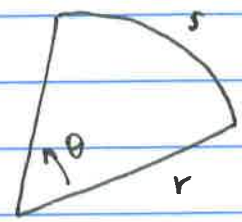
What about the angular speed:

$$\omega = \frac{2\pi \text{ rad}}{60 \text{ sec}} = \frac{\pi}{30} \text{ rad/sec}$$

We may then deduce that the clock hand travels

$$\frac{\cancel{\pi}}{3} \frac{\text{cm}}{\cancel{\text{sec}}} \cdot \frac{30}{\cancel{\pi}} \frac{\cancel{\text{sec}}}{\text{rad}} = \boxed{10 \text{ cm/rad.}}$$

Area of a sector of a circle



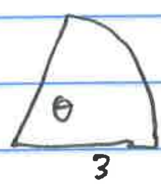
Area $= \frac{1}{2} r^2 \theta$

Idea: a sector is like a triangle with one side made out of an arc. Area of a triangle is $\frac{1}{2} bh$.

$\theta = \frac{s}{r}$, so $\frac{1}{2} r^2 \theta = \frac{1}{2} r^2 \frac{s}{r} = \frac{1}{2} rs$.

It's not a proof by any means, but it does help w/ intuition.

Ex.



$\theta = \frac{\pi}{6}$, $r = 3$ Area $= \frac{1}{2} \cdot 3^2 \cdot \pi/6 = \boxed{\frac{3\pi}{4}}$

Ex. A sprinkler at a golf course sprays water 70 ft and rotates through an angle of $2\pi/3$ rad. Find the area of grass it waters.

$$\text{Area} = \frac{1}{2} r^2 \theta = \frac{1}{2} (70)^2 \left(\frac{2\pi}{3}\right) = \frac{1}{2} (4900) \frac{2\pi}{3} = \frac{4900\pi}{3}$$
$$\approx 5131 \text{ ft}^2.$$

Degrees: divide the circle into 360 equal "slices" each slice has measure 1 degree, written $1°$.

Covert: degrees to radians and vice versa.

Convert to rad: 120°, -20°

to deg: $\frac{-7\pi}{12}$, $\frac{5\pi}{4}$

minutes and seconds.

Convert to decimal: $54^\circ 45'$

$-135^\circ 36''$

Find arc length: $r = 15$ in, $\theta = 120^\circ$

Find central angle in deg: $r = 80$ km, $s = 150$ km.

$$\frac{15}{8} = 1.875 \text{ rad} \times \frac{180}{\pi}$$

$$= 107.43^\circ$$

Ex. A car is moving 65 mph, diameter of wheel is 2 ft.

Find rev/min:

$$65 \frac{\cancel{mi}}{\cancel{hr}} \cdot \frac{1 \cancel{hr}}{60 \text{ min}} \cdot \frac{1 \text{ rev}}{2\pi \cancel{ft}} \cdot \frac{5280 \cancel{ft}}{\cancel{mi}} = \frac{65(5280)}{60(2\pi)} \frac{\text{rev}}{\text{min}}$$

$$\approx \boxed{910 \frac{\text{rev}}{\text{min}}}$$

1 rev = 2π rad, and the arc length (dist traveled) is $s = r\theta = 1 \text{ ft}(2\pi) = 2\pi$ ft

Find the angular speed of the wheel $\left(\frac{\text{rad}}{\text{min}}\right)$

$$910 \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} = \boxed{1820\pi \frac{\text{rad}}{\text{min}}}$$

Ex. Prove the Pythagorean Thm.