Statement of Research Interests

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My primary research interests are in the theory of general connections on fiber bundles, a branch of Differential Geometry. In particular, I have characterized the general connections which admit a system of parallel transport along the base manifold [3].

Theorem 1 *A general connection on a smooth fiber bundle* $\pi : E \twoheadrightarrow M$ *admits a system of parallel transport if and only if it is uniformly vertically bounded.*

Most work in the theory of general connections assumes that the connection admits parallel transport, so this is a fundamental result. In fact, in 1950 Charles Ehresmann [2] included a horizontal path lifting property (see *infra*) in his definition of a general connection to ensure that one admits parallel transport. The above theorem is proved by characterizing the general connections with this property. No other complete classification has been given, to my knowledge.

The methods developed in proving this theorem have also given further insight into the nature of linear connections on vector bundles. Linear connections are representative members of a larger class of general connections called *vertically constant*. These are the most tractable family of general connections on fiber bundles.

Current Work

Connections were originally studied on tangent bundles as a way to generalize parallel translation in a vector space. They are so named because the parallel transport that they define along the base manifold M effectively *connects* tangent spaces over distant points in M. This idea extends to fibers of arbitrary fiber bundles.

Let $\pi : E \twoheadrightarrow M$ be a smooth fiber bundle with tangent bundle $\pi_T : TE \twoheadrightarrow E$. If one thinks of the base manifold M as horizontal and the fibers of E as vertical, then there is a natural notion of vertical tangent vectors in TE. The *vertical bundle* $\pi_{\mathscr{V}} : \mathscr{V} \twoheadrightarrow E$ consists of all vectors in TE that are tangent to the fibers of E. This vertical bundle is *natural*: it is a global subbundle of the tangent bundle TE over E. In general, there is no globally well-defined way to define horizontal tangent vectors. A general connection on E is a choice of horizontal subbundle.

Definition 2 A *general connection* on a fiber bundle $\pi : E \twoheadrightarrow M$ is a subbundle $\mathscr{H}$ of the tangent bundle TE that is complementary to the vertical bundle, so that $TE = \mathscr{H} \oplus \mathscr{V}$.

Parallel transport is defined by lifting paths in the base manifold M to the total space E in such a way that the lifted path's tangent vectors always lie in $\mathscr{H}$. Let $\gamma : I \to M$ be a path in M with $\gamma(0) = p$ and $\gamma(1) = q$. A horizontal lift of γ is a path $\overline{\gamma} : I \to E$ whose velocity lift $\dot{\overline{\gamma}} : I \to TE$ is a path in $\mathscr{H}$; that is, the velocity vector field along $\overline{\gamma}$ is horizontal.

The following property was included by Ehresmann [2] in his definition of a general connection in 1950.

Definition 3 A general connection $\mathscr{H}$ is said to have the *horizontal path lifting property* if and only if, for every path $\gamma : I \to M$ and every $v \in E_p$, there exists a unique horizontal lift $\overline{\gamma} : I \to E$ such that $\overline{\gamma}(0) = v$ and $\overline{\gamma}(1) \in E_q$.

In other words, the horizontal lifts of every path in M extend over the entire interval $I = [0, 1]$.

If a general connection $\mathscr{H}$ has the horizontal path lifting property, then the horizontal lifts of each path γ foliate the pullback bundle $\gamma^* E$ over I. Each such foliation defines a diffeomorphism $\mathcal{P}_\gamma$ between the fibers E_p and E_q called *parallel transport along* γ. The collection of all such diffeomorphisms is called a system of parallel transport in E.

Conversely, a given system of parallel transport in E defines a connection on E by determining the horizontal subspaces in each fiber of TE. Therefore a connection admits parallel transport in E if and only if it has the horizontal path lifting property.

Since $\mathscr{H}$ and $\mathscr{V}$ are complementary subbundles, the only way that a general connection can fail to have horizontal path lifting is if the horizontal spaces become asymptotic to the vertical spaces along a fiber of E. This fact motivated the following definition in [3].

Definition 4 A general connection $\mathscr{H}$ on a fiber bundle $\pi : E \twoheadrightarrow M$ is said to be *uniformly vertically bounded* if and only if $\mathscr{H}_v$ is bounded away from $\mathscr{V}_v$, uniformly along fibers of E.

In [3], I showed that uniform vertical boundedness is both necessary and sufficient for a general connection on a fiber bundle to have the horizontal path lifting property.

The main tool used in the proof is a collection of $m = \min\{n, k\}$ nonzero *Wong angles* between the n-dimensional horizontal and k-dimensional vertical subspaces in each fiber of TE [3, 4]. These are measured by putting a fixed auxiliary Riemannian metric on the model fiber F, then using it to form local product metrics on E. The Levi-Civita connection for the product metric is then used to compare Wong angles along the fibers of E. In terms of Wong angles, a general connection is then uniformly vertically bounded if and only if the Wong angles are bounded away from zero, uniformly along fibers of E.

A general connection can be described locally by a vertical-vector-valued 1-form on the base manifold M. Suppose $(U_\alpha, \varphi_\alpha)$ is a trivializing chart on M for E, and let $E_\alpha = (\varphi_\alpha^{-1})^* E \cong U_\alpha \times F$. The restricted tangent bundle is $T(E_\alpha) = TU_\alpha \times TF = TU_\alpha \oplus \mathscr{V}_\alpha$. Let $\mathcal{V}$ be the unique projection of TE onto $\mathscr{V}$ with $\ker \mathcal{V} = \mathscr{H}$. The *local Christoffel form* for $\mathscr{H}$ on U_α is defined by

$$\Gamma^\alpha_\sigma : \mathfrak{X} U_\alpha \to \mathscr{V}_\alpha : X \mapsto -\mathcal{V}_\sigma(\sigma_* X),$$

where $\sigma \in \Gamma E_\alpha$.

These Christoffel forms measure the failure of $\mathscr{H}$ to be locally trivial over U_α. If $\theta_p \in [0, \pi/2]$ is the lower bound for the Wong angles along the fiber E_p, then the norm of the Christoffel form along E_p is defined by

$$\|\Gamma^\alpha\| = \tan(\pi/2 - \theta_p).$$

A connection $\mathscr{H}$ is then uniformly vertically bounded if and only if $\|\Gamma^\alpha\|$ is bounded above along each fiber of E.

Putting it all together, I have obtained the following theorem.

Theorem 5 *Let $\mathscr{H}$ be a general connection on a smooth fiber bundle $\pi : E \twoheadrightarrow M$ with model fiber F. The following are equivalent.*

i.) $\mathscr{H}$ is uniformly vertically bounded (UVB);

ii.) $\mathscr{H}$ has the horizontal path lifting (HPL) property;

iii.) $\mathscr{H}$ admits a system of parallel transport in E, along M;

iv.) There exists a Riemannian metric on F such that the Christoffel forms Γ^α are bounded above, uniformly along fibers of E.

Ancillary Questions and Future Program

There is more work to be done in the theory of uniformly vertically bounded (UVB) connections on fiber bundles. I've shown that UVB connections are Whitney stable, but neither Schwartz nor homotopy stable. Using the

Christoffel forms, it is possible to define a covariant derivative for UVB connections. I am also investigating what role, if any, the Wong angles play in determining the curvature and holonomy of a general connection.

The usage of Wong angles in this development has given new insight into the nature of general connections on fiber bundles. It is known that if $\mathscr{H}$ is a linear connection on a vector bundle E, then the Wong angles for $\mathscr{H}$ are constant along fibers of E. This generalizes to define a class of general connections on fiber bundles that contains linear connections.

Definition 6 A general connection $\mathscr{H}$ on a fiber bundle $\pi : E \twoheadrightarrow M$ is called *vertically constant* if and only if its Wong angles are constant along each fiber of E.

If $\mathscr{H}$ is vertically constant, then the Wong angles along any section of E determine the connection on all of E. This fact should make it possible to obtain results for vertically constant connections that would then generalize to UVB connections.

There is also work to do in the theory of nonlinear connections on vector bundles. In particular, Del Riego and Parker [1] have studied nonlinear connections on the tangent bundle *via* special associated second-order differential equations that provide the natural geodesics for these connections. The same authors also began studying Jacobi fields for nonlinear connections on tangent bundles. These results should be extended at least to vector bundles.

References

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[3] P. E. Parker and J. M. Ryan, Horizontal Path Lifting for General Connections, DGS Preprint PR2, tp + 6 pp. http://arxiv.org/abs/1105.4554

[4] Y.-C. Wong, Differential geometry of Grassmannian manifolds, *Proc. Nat. Acad. Sci USA* **57** (1967) 589–594.