Horizontal Path Lifting for General Connections

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Fiber Bundles

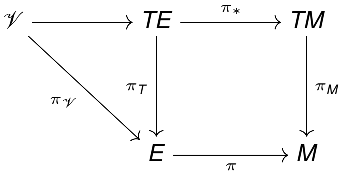
A smooth *fiber bundle* is a quadruple (E, π, M, F) such that

- M is a smooth n-manifold called the *base*
- F is a smooth k-manifold called the *model fiber*
- E is a smooth $(n+k)$-manifold called the *total space*
- $\pi : E \twoheadrightarrow M$ is a smooth surjection such that $\pi^{-1}(p) \cong F$ for all $p \in M$

$E_p := \pi^{-1}(p)$ is called the *fiber* over p.

General Connections

The *vertical bundle* $\pi_{\mathscr{V}} : \mathscr{V} \twoheadrightarrow E$ is the kernel of the induced tangent map $\pi_* : TE \twoheadrightarrow TM$, $\mathscr{V} := \ker \pi_*$. It is a subbundle of the tangent bundle $\pi_T : TE \twoheadrightarrow E$.



A *general connection* on $\pi : E \twoheadrightarrow M$ is a subbundle $\mathscr{H}$ of the tangent bundle $\pi_T : TE \twoheadrightarrow E$ that is complementary to the vertical bundle $\mathscr{V}$, so that

$$TE = \mathscr{H} \oplus \mathscr{V}.$$

Pullback Bundles

If $f : N \to M$ is a smooth map of manifolds and $\pi : E \twoheadrightarrow M$ is a fiber bundle over M, then the *pullback bundle* $f^*\pi : f^*E \twoheadrightarrow N$ has fibers

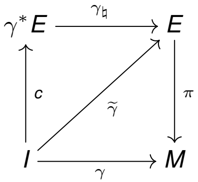
$$(f^*\pi)^{-1}(p) := \pi^{-1}(f(p))$$

over N, and the following diagram commutes.

$$\begin{array}{ccc} f^*E & \xrightarrow{f_\natural} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[f]{} & M \end{array}$$

Path Lifts

Let $\gamma : I \to M$ be a path in M with $\gamma(0) = p$ and $\gamma(1) = q$. Consider the pullback bundle $\gamma^*\pi : \gamma^*E \twoheadrightarrow I$. A *lift* $\widetilde{\gamma} : I \to E$ of γ to E is the pushforth of a section of $\gamma^*\pi$, $\widetilde{\gamma} = \gamma_\natural c$.



The space of all lifts of γ is denoted by $\Gamma_\gamma E$.

A lift $\overline{\gamma}$ of γ is called *horizontal* if $\dot{\overline{\gamma}}(t) \in \mathscr{H}_{\overline{\gamma}(t)}$ for all $t \in I$.

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The HPL Property

A general connection $\mathscr{H}$ has the *horizontal path lifting* (HPL) property if for every path $\gamma : I \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$ and initial value $v \in E_p$, there exists a unique horizontal lift $\overline{\gamma}$ such that $\overline{\gamma}(0) = v$ and $\overline{\gamma}(1) \in E_q$.

In other words, all horizontal lifts of paths in M extend over all of I.

Or in yet other words, the horizontal lifts of γ foliate the pullback bundle $\gamma^* E$ such that every leaf of the foliation meets every fiber of $\gamma^* E$. Such a foliation is called *complete* by some authors.

Connections with HPL are called *Ehressmann* connections, as he actually included HPL in his definition of a connection.

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Parallel Transport

Horizontal lifts of paths in M are important geometrically because they can be used to “connect” the fibers of E along γ in the following sense:

If $\mathscr{H}$ has HPL, then the horizontal lifts of γ determine a complete foliation of the pullback bundle $\gamma^* E$, yielding a diffeomorphism

$$\mathcal{P}_\gamma : E_p \cong E_q.$$

The diffeomorphism $\mathcal{P}_\gamma$ is called *parallel transport* along γ, and can be used to “move” elements in the fiber over p to the fiber over q.

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What Can Go Wrong?

The condition $\dot{\tilde{\gamma}}(t) \in \mathscr{H}_{\tilde{\gamma}(t)}$ can be thought of as a generalized differential equation for which we are charged with finding the integral curves.

In ODE theory, the only way that the solution (integral curve) of an IVP will fail to extend over all of I is if the derivative becomes unbounded, or "runs off to infinity."

The analog here is that $\mathscr{H}$ could become asymptotic to $\mathscr{V}$ along an end of a fiber of E, or as one "runs off to infinity" of the fiber.

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A 1-Dimensional Example
due to P. E. Parker (2011)

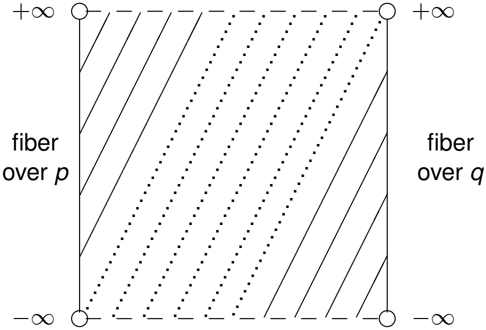

Figure: Horizontal lifts of a path c from p to q in M. Infinity of the fibers has been brought into the finite plane by a compression such as $y \mapsto \tanh y$. The solid lines begin in one fiber and go away to infinity. The dotted lines do not intersect either fiber. Note that no horizontal lift of c reaches from over p to over q.

Uniform Vertical Boundedness

To remove this obstacle we require that the horizontal spaces $\mathscr{H}_v$ are bounded away from the vertical spaces $\mathscr{V}_v$, uniformly along fibers of E. Such an $\mathscr{H}$ is said to be *uniformly vertically bounded* (UVB). This idea is due to Parker (2011).

For this property to be useful, we need a way to measure whether a given $\mathscr{H}$ is UVB or not.

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Wong Angles

Each fiber T_vE of TE looks like a copy of $\mathbb{R}^{n+k}$; the horizontal and vertical spaces $\mathscr{H}_v$ and $\mathscr{V}_v$ being an n- and k-dimensional hyperplane, respectively.

Wong (1967) showed that, given an inner product on $\mathbb{R}^{n+k}$, one can find $m = \min\{n, k\}$ nonzero *Wong angles* between any complementary n- and k-planes.

Denote the Wong angles in the fiber T_vE by $\theta_i(v)$, $i = 1, \ldots, m$, and the smallest Wong angle in T_vE by $\theta(v)$.

Comparing Wong Angles

Let g be an auxiliary Riemannian metric on E, and ∇^g its Levi-Civita connection. It is well known that ∇^g has HPL, thus determines a system of parallel transport $\mathcal{P}^g$.

Fix $v \in E_p$, and let $c : w \to v$ be a path in E_p. Let $\theta_v(w, c)$ denote the (smallest) Wong angle between the n-plane $\mathcal{P}^g_c(\mathscr{H}_w)$ and the k-plane $\mathcal{P}^g_c(\mathscr{V}_w)$ as measured in $T_v E \cong \mathcal{P}^g_c(T_w E)$.

The connection $\mathscr{H}$ is then UVB if and only if

$$\inf_{w \in E_p} \left\{ \inf_{c : w \to v} \theta_v(w, c) \right\} \geq \varepsilon > 0.$$

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HPL iff UVB

The UVB condition turns out to be exactly what is needed to determine whether a general connection has HPL or not.

Theorem

Let E be a fiber bundle of fiber-dimension k over an n-dimensional manifold M, $\mathscr{H}$ a general connection on E, and $\gamma : I \to M$ a path with $\gamma(0) = p$ and $\gamma(1) = q$. For every $v \in E_p$, there exists a unique horizontal lift $\overline{\gamma} : I \to E$ such that $\overline{\gamma}(0) = v$ and $\overline{\gamma}(1) \in E_q$ if and only if $\mathscr{H}$ is UVB.

Sketch of a Proof

One direction is fairly simple:

If $\mathscr{H}$ has HPL, then the horizonal lifts of γ foliate the pullback bundle $\gamma^* E$. Moreover, every leaf of the foliation $\mathfrak{F}$ meets every fiber of $\gamma^* E$. Since $\gamma_\natural T(\mathfrak{F}) \subseteq \mathscr{H}$ and γ was arbitrary, the connection $\mathscr{H}$ is UVB.

After all, UVB was only defined in the first place *because* of this fact.

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Sketch of a Proof

The other direction requires more work:

Since I is compact, $\mathrm{im}(\gamma)$ is compact in M, and we may assume that $\mathrm{im}(\gamma)$ is contained in a single trivializing chart $U \subseteq M$ for both E and TM.

Now the tangent bundle to E over U looks like

$$\begin{aligned} T(E_U) &\cong T(U \times F) \\ &\cong TU \times TF \\ &\cong \mathscr{B}_U \oplus \mathscr{V} \end{aligned}$$

where $\mathscr{B}_U := TU$ is the (purely local) *basal* subbundle of $T(E_U)$.

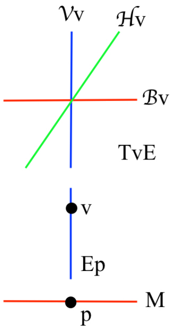
Sketch of a Proof

Let $W \subseteq F$ be a trivializing chart for TF. Then $E_{(U,W)} \cong U \times W$ can be given local coordinates (x, y). Use these coordinates to induce local coordinates (x, y, X, Y) on $T(E_{(U,W)})$.

Now vectors in $\mathscr{B}_U$ are of the form (x, y, X) and vectors in $\mathscr{V}_W$ are of the form (x, y, Y).

Sketch of a Proof

The relationship between $\mathscr{B}_U$, $\mathscr{V}_W$, and $\mathscr{H}_{(U,W)}$ in a single fiber of $T_v(E_{(U,W)})$ can be visualized by this schematic picture:



Sketch of a Proof

Let c be a section of the pullback bundle $\gamma^* E$ with $c(0) = v$. Since I is contractible, $\gamma^* E$ is a trivial bundle, and we may identify all of its fibers with a single copy of F.

Thus we may think of $c : I \to F$ as a path in F with $c(0) = v$. As before, we may assume that $\mathrm{im}(c)$ is contained in a single trivializing chart W, and that $\mathrm{im}(c)$ has no self-intersections.

Furthermore, assume that the corresponding lift $\gamma_\natural c = (\gamma, c)$ is horizontal. That is,

$$(\gamma_\natural c)^{\mathrm{C}}(t) = (\gamma, c, \dot{\gamma}, \dot{c})(t) \in \mathscr{H}_{\gamma_\natural c(t)}$$

for all $t \in I$ such that c is defined.

Sketch of a Proof

It remains to prove that, under these conditions, c extends over all of I.

To do so, reconsider our three subbundles of TE over $\mathrm{im}(\gamma)$. Vectors in $\mathscr{B}_U$ along γ look like

$$\dot{\gamma}_i(t)X^i,$$

and vectors in $\mathscr{H}_{(U,W)}$ differ from those in $\mathscr{B}_U$ only by translations in $\mathscr{V}_W$:

$$\dot{\gamma}_i(t)X^i + f_j(y)Y^j.$$

The lift $\gamma_\natural c$ can be written in local coordinates as

$$\dot{\gamma}_i(t)X^i + \dot{c}_j(t)Y^j.$$

Sketch of a Proof

This gives a differential equation for c. The lift $\gamma_\natural c$ is horizontal if and only if c solves the IVP

$$\dot{c}(t) = f(c(t)), \quad c(0) = v.$$

Since $\mathscr{H}$ is UVB, then f is bounded uniformly on F. The FEUT and Extension Theorem of standard ODE theory apply and give us that c exists, is unique, and extends over all of I. $\square$

Reflections

Parker (2011) originally proved this theorem on tangent bundles $E = TM$ directly. He was able to take advantage of many nice features of tangent bundles to give a much simpler, and more simple-minded, proof.

His theorem was first extended to vector bundles by using the tangent-bundle proof as a model. The next extension was to bundles with parallelizable fibers, with the final extension to arbitrary fiber bundles.

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Reflections

Ehressmann (1950) showed that if F is compact, then every connection on E has HPL.

Kolár, Michor, and Slovák (1993) gave a sufficient condition for a connection $\mathscr{H}$ to have HPL that turns out to be very much in the spirit of PR's proof. Upon comparison, the $f(c(t))$ in PR's DE is actually the vector field determined by the Christoffel form $\Gamma^{\gamma}(c(t))$ along γ. This deserves further investigation.

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