Geometry of Horizontal Bundles and Connections

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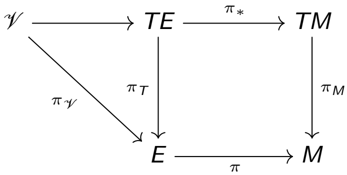
The Vertical Bundle

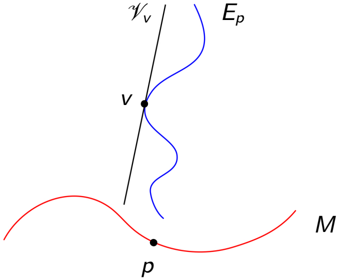
- Let $\pi : E \twoheadrightarrow M$ be a smooth fiber bundle.
- The *vertical bundle* $\pi_{\mathscr{V}} : \mathscr{V} \twoheadrightarrow E$ over E is the kernel of the induced tangent map $\pi_* : TE \twoheadrightarrow TM$, $\mathscr{V} := \ker \pi_*$. It is a (vector) subbundle of the tangent bundle $\pi_T : TE \twoheadrightarrow E$.



- The vertical bundle $\mathscr{V}$ is natural: there is only one.

The Vertical Bundle

- Over any point $v \in E$, the vertical fiber $\mathscr{V}_v$ consists of all of the vectors in $T_v E$ that are tangent to the fiber E_p of E containing v.



- Therefore $\mathscr{V}_v \cong T_v F$, where F is the model fiber of $\pi : E \twoheadrightarrow M$.

Horizontal Bundles

- A *horizontal bundle* on $\pi : E \twoheadrightarrow M$ is a (vector) subbundle $\mathscr{H}$ of the tangent bundle $\pi_T : TE \twoheadrightarrow E$ that is complementary to the vertical bundle $\mathscr{V}$, so that
$$TE = \mathscr{H} \oplus \mathscr{V}.$$
- Horizontal bundles are *not* natural in general; hence the plurality.

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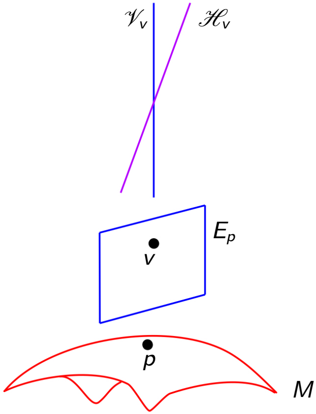
Horizontal Bundles

- In each tangent space $T_v E$, an admissible horizontal space $\mathscr{H}_v$ is an n-plane that is complementary to the vertical space $\mathscr{V}_v$ in that fiber.



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Horizontal Bundles

A horizontal bundle is completely determined by any one of the following:

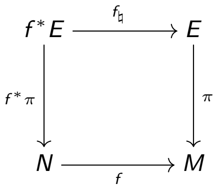
- A section Γ of the horizontal Grassmann bundle $G_H(TE) \twoheadrightarrow E$.
- A section Γ of the first jet prolongation $J^1E \twoheadrightarrow E$ of E.
- A projection $\mathcal{V} : TE \twoheadrightarrow \mathscr{V}$, whence $\mathscr{H} := \ker \mathcal{V}$.
- A projection $\mathcal{H} : TE \to TE$ satisfying $\ker \mathcal{H} = \mathscr{V}$, whence $\mathscr{H} := \operatorname{im} \mathcal{H}$.

Pullback Bundles

- If $f : N \to M$ is a smooth map of manifolds and $\pi : E \twoheadrightarrow M$ is a fiber bundle over M, then the *pullback bundle* $f^*\pi : f^*E \twoheadrightarrow N$ has fibers

$$(f^*\pi)^{-1}(p) \cong \pi^{-1}(f(p))$$

over N, and the following diagram commutes.



Pullback Bundles

- Pullback is a functor: it preserves the vertical bundle.

$$\begin{array}{ccccc}
(f_\natural)^*(\mathscr{V}E) & \xrightarrow{\cong} & \mathscr{V}(f^*E) & \xrightarrow{f_{\natural *}} & \mathscr{V}E \\
& \searrow^{f_\natural^*\pi_{\mathscr{V}}} & \downarrow \pi_{\mathscr{V}} & & \downarrow \pi_{\mathscr{V}} \\
& & f^*E & \xrightarrow{f_\natural} & E \\
& & \downarrow f^*\pi & & \downarrow \pi \\
& & N & \xrightarrow[f]{} & M
\end{array}$$

Parallel Transport

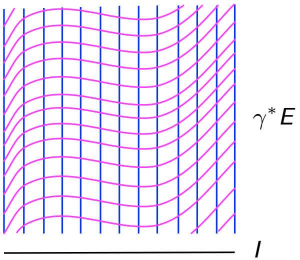
- Let $\pi : E \twoheadrightarrow M$ be a fiber bundle with horizontal bundle $\mathscr{H} \leq TE$, and let $\gamma : I = [0,1] \to M$ be a path in M with $\gamma(0) = p$ and $\gamma(1) = q$.
- The vertical projection $\mathcal{V}$ associated to $\mathscr{H}$ pulls back to a vertical projection on the pullback bundle $\gamma^* E \twoheadrightarrow I$.

$$\widetilde{\mathcal{V}} : T(\gamma^* E) \twoheadrightarrow \mathscr{V}(\gamma^* E) : v \mapsto \gamma_\natural^* \mathcal{V}(\gamma_{\natural *} v).$$

- $\widetilde{\mathscr{H}} := \ker \widetilde{\mathcal{V}}$ is a 1-dimensional subbundle of $T(\gamma^* E)$.

Parallel Transport

- Since $\widetilde{\mathscr{H}}$ is a line bundle, it is integrable. Its integral curves foliate the total space $\gamma^* E$.



- We call this foliation $\mathcal{P}_\gamma$, and denote the unique leaf emanating from $v \in E_p$ by $\mathcal{P}_\gamma v$.

Parallel Transport

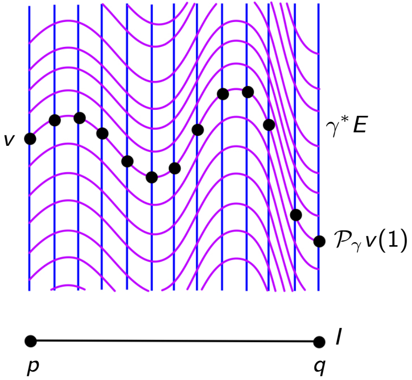
- If every leaf of $\mathcal{P}_\gamma$ meets every fiber of $\gamma^* E$, we say that γ has *horizontal lifts*. In this case, $\mathcal{P}_\gamma$ determines a diffeomorphism between the fibers over γ.

$$\mathcal{P}_\gamma(t) : E_p \to E_{\gamma(t)} : v \mapsto \mathcal{P}_\gamma v(t)$$

- In particular, the map $v \mapsto \mathcal{P}_\gamma v(1)$ is a diffeomorphism between the fibers E_p and E_q over the points which γ "connects" in the base.
- The map(s) $\mathcal{P}_\gamma$ is called *parallel transport along* γ.

Parallel Transport

- Parallel transport along a path γ, with initial value $\overline{\gamma}(0) = v$.



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Parallel Transport

- If every path γ in M has horizontal lifts, then we say that $\mathscr{H}$ has *horizontal path lifting*, or HPL.
- If $\mathscr{H}$ has HPL, then the parallel transport it induces can be used to connect the fibers over any two points in the same path component of M.
- An *Ehresmann connection*, or simply a connection, is defined to be a horizontal bundle with HPL.

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A Natural Question

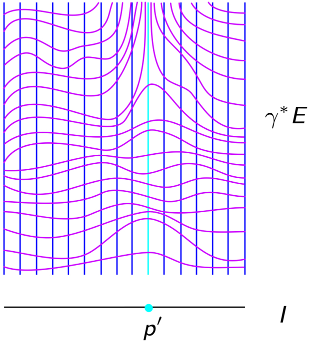
- Ehresmann recognized that HPL is a nontrivial property if the fibers of E are not compact. This is evidenced by his inclusion of HPL in his definition of a connection.
- This begs the natural question:

Question

Which horizontal bundles actually determine connections on E?

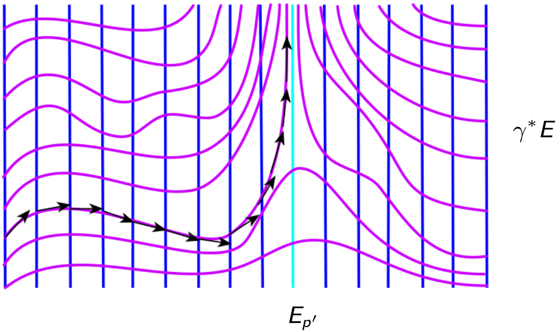
The Problem

- To answer this fundamental question, one must first understand what could possibly go wrong.
- The problem is that the leaves of the horizontal foliation along a given curve may become asymptotic to a fiber of $\gamma^* E$, and "run off to infinity."



The Problem

- Looking more closely at this foliation, we see that the tangent vectors along the "bad" leaves asymptotically approach vertical vectors along the fiber $E_{p'}$.



The Solution

- To ensure that a horizontal bundle $\mathscr{H}$ is a connection, we must require that the velocity fields along the leaves of $\mathcal{P}_\gamma$ be bounded away from the vertical space for every path γ.
- This is achieved by demanding that the horizontal bundle $\mathscr{H}$ is *uniformly vertically bounded*:

Definition

A horizontal bundle $\mathscr{H}$ is *uniformly vertically bounded*, or UVB, if and only if the horizontal spaces $\mathscr{H}_v$ are bounded away from the vertical spaces $\mathscr{V}_v$, uniformly along the fibers of E.

Uniform Vertical Boundedness

- For this to be useful, one needs a way to measure the distance between $\mathscr{H}$ and $\mathscr{V}$ along a fiber of E.
- Wong (1967) used a collection of principal angles to study the geometry of Grassmann manifolds.
- Parker (2011) noticed that since a horizontal bundle is a section of a special Grassmann bundle, these angles could be used to compare a horizontal bundle to the vertical bundle.
- He then used these Wong angles to prove that HPL is equivalent to UVB when $E = TM$.

HPL if and only if UVB

- The main result of my dissertation is the extension of Parker's theorem to the general fiber bundle case.

Theorem

Let $\pi : E \twoheadrightarrow M$ be a smooth fiber bundle. A horizontal bundle $\mathscr{H}$ on E is a connection if and only if $\mathscr{H}$ is UVB.

Wong Angles

- Let U_α be a trivializing chart at $p \in M$, so that $E_\alpha = U_\alpha \times F$. The tangent bundle TE_α decomposes as

$$\begin{aligned} TE_\alpha &= TU_\alpha \times TF \\ &= \mathscr{B}_\alpha \times \mathscr{V}_\alpha. \end{aligned}$$

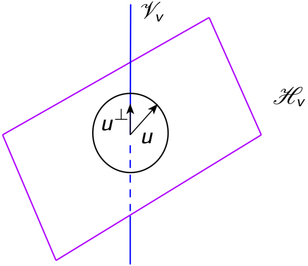
- Let g_F be a Riemannian metric on F and g_α the Euclidean metric on U_α. Then the product metric $g = g_\alpha \times g_F$ makes $E_\alpha = U_\alpha \times F$ a Riemannian manifold.

- This makes $TE_\alpha = \mathscr{B}_\alpha \oplus \mathscr{V}_\alpha$ an orthogonal sum with respect to g.

Wong Angles

- Let $u \in \mathscr{H}_v$ and let $u^\perp \in \mathscr{V}_v$ be its orthogonal complement.
- The *Wong angles* between $\mathscr{H}_v$ and $\mathscr{V}_v$ are the non-zero stationary values of

$$\theta_v(u) := \cos^{-1}\left(\frac{|g_v(u, u^\perp)|}{\|u\|\,\|u^\perp\|}\right).$$



Comparing Wong Angles

- The Levi-Civita connection for g induces parallel transport $\mathcal{P}^g$ along each fiber E_p of E_α.
- Let $\delta : I \to E_p$ be a path in E_p with $\delta(0) = w$ and $\delta(1) = v$. To compare the Wong angles in T_wE and T_vE, transport $\mathscr{H}_w$ and $\mathscr{V}_w$ to sit over v.
- Define $\theta_v(w, \delta)$ to be the smallest Wong angle between the n- and k-planes $\mathcal{P}^g_\delta(\mathscr{H}_w)$ and $\mathcal{P}^g_\delta(\mathscr{V}_w)$ in T_vE, as measured by the inner product g_v.
- The Wong angles along the fiber E_p are bounded below by

$$\theta_p := \inf_{w \in E_p} \left\{ \inf_{\delta : w \mapsto v} \theta_v(w, \delta) \right\} \geq 0.$$

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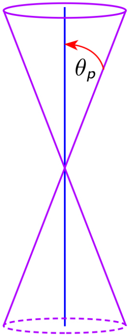
UVB *via* Wong Angles

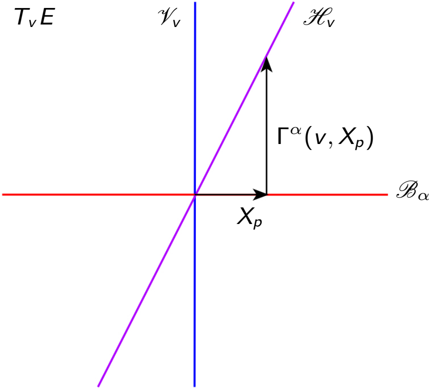
- A horizontal bundle is UVB if and only if the Wong angles θ_p are bounded away from zero along every fiber E_p, $p \in M$.

$$\theta_p \geq \varepsilon_p > 0, \quad p \in M.$$



Local Christoffel Forms

- In every fiber T_vE, the horizontal spaces have components in both the basal and vertical directions.



Local Christoffel Forms

- The *local Christoffel forms* for E_α are maps

$$\Gamma^\alpha : \Gamma E_\alpha \times \mathfrak{X}_\alpha \to \Gamma \mathscr{V}_\alpha$$

  defined by

$$\Gamma^\alpha(\sigma, X)(p) = -\mathcal{V}_{\sigma_p}(p, \sigma_p, X_p, 0).$$

- Fixing $\sigma \in \Gamma E_\alpha$ makes $\Gamma^\alpha(\sigma, \ ) =: \Gamma^\alpha_\sigma$ a $\mathscr{V}$-valued 1-form on U_α, along σ.

$$\Gamma^\alpha_\sigma : \mathfrak{X}_\alpha \to \mathscr{V}_\sigma$$

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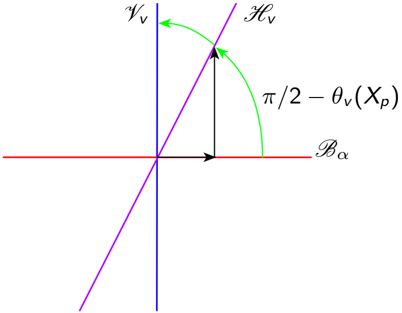
LCFs and Wong Angles

- The local Christoffel forms are bounded in terms of the Wong angles.

$$\|\Gamma^{\alpha}\| \leq \tan(\pi/2 - \theta_p)$$

along the fiber E_p.



UVB *via* Christoffel Forms

- This implies that $\mathscr{H}$ is UVB if and only if the local Christoffel forms are bounded above, along every fiber of E.
- In particular, along any path $\gamma : I \to U_\alpha$, the horizontal spaces are given by

$$\mathscr{H}_v = (\gamma, c, \dot{\gamma}, \Gamma^\alpha(c, \dot{\gamma})).$$

- This yields a differential equation for c along γ,

$$c'(t) = \Gamma^\alpha(c(t), \dot{\gamma}(t)).$$

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HPL if and only if UVB

- Applying the standard FEUT (and friends), we obtain:
- If Γ^α is bounded, then the differential equation $c' = \Gamma^\alpha(c, \dot{\gamma})$ has a unique solution for every initial value $c(0) = v$, and the solution extends over the entire domain of γ.
- Since Γ^α is bounded along an arbitrary path γ if and only if $\mathscr{H}$ is UVB, then $\mathscr{H}$ has HPL if and only if it is UVB.
- Thus UVB is exactly what is needed to ensure that a horizontal bundle connects the fibers of E *via* parallel transport, and hence is a connection.

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Ancillary Results

- The local Christoffel forms can be used to characterize many geometric properties of horizontal bundles, and illustrate the differences between horizontal bundles and connections.

Covariant Derivatives

- If E is a vector bundle, then any horizontal bundle defines a covariant derivative, $\nabla : \mathfrak{X} \times \Gamma E \to \Gamma E$.

  $$\nabla_X \sigma := \mathcal{K} \circ \mathcal{V}(\sigma_* X),$$

  where $\mathcal{K} : \mathscr{V} \to E$ is fiber-isomorphism.

- Over a trivializing chart U_α, the covariant derivative is given by

  $$\nabla_X \sigma = X^j \frac{\partial \sigma^i}{\partial x^j} \xi_i - (\mathcal{K}\Gamma^\alpha_\sigma X)^i \xi_i,$$

  where ξ is a constant frame field for the fibers of E_α.

- Therefore the local Christoffel forms are just lifted versions of the classical Christoffel symbols.

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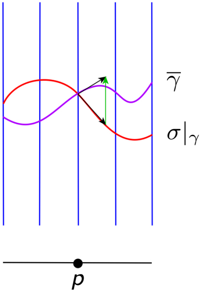
Covariant Derivatives

- Suppose $p \in U_\alpha$ and let γ be the integral curve of X passing through p. Let $\overline{\gamma}$ be the horizontal lift of γ with initial value $\overline{\gamma}(p) = \sigma(p)$.
- The covariant derivative measures the failure of σ to be horizontal along the integral curves of X. Thus $(\nabla_X \sigma)(p) = \mathcal{K}(\dot{\overline{\gamma}}(p) - \dot{\sigma}|_\gamma(p))$.



Induced Quasisprays

- Now let $\mathscr{H}$ be a horizontal bundle on $TM \twoheadrightarrow M$.
- $\mathscr{H}$ induces a quasispray (or semi-spray) on M,

$$Q(v) := \pi_*|_{\mathscr{H}_v}^{-1}(v).$$

  We write $\mathscr{H} \vdash Q$ to denote this relationship.
- This can be written locally in terms of the Christoffel forms,

$$Q(v) = (p, v, v, \Gamma^\alpha(v, v)).$$

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Torsion

- Many horizontal bundles induce the same quasispray. In fact, the space of horizontal bundles fibers over the space of quasisprays.
- Del Riego and Parker defined the *torsion* of a horizontal bundle to be that which varies over these fibers.
- The torsion-free horizontal bundles are the ones which satisfy

$$\Gamma^\alpha(v, X) = \Gamma^\alpha(X, v)$$

for all $X \in \mathfrak{X}$ and all $v \in TM$.

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Understanding Torsion?

- Thus, the torsion-free horizontal bundles are the ones whose local Christoffel forms are all symmetric.
- Another way to say this is that the torsion-free horizontal bundles respect the π_*-structure on TTM: they induce the same horizontal bundle over both structures π_T and π_*.
- Thus torsion measures the failure of $\mathscr{H}$ to be invariant under the change of bundle structures from π_T to π_*:

$$\mathfrak{I}\mathscr{H} = \mathscr{H}.$$

Geodesics

- A path $\gamma : I \to M$ is a *geodesic* of $\mathscr{H}$ if and only if its velocity lift is horizontal (or $\mathscr{H}$-parallel): $\dot{\gamma} = \overline{\gamma}$.
- This can be written in terms of:
  - the covariant derivative:
    A path γ is an$\mathscr{H}$-geodesic if and only if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$;
  - the quasispray $Q \dashv \mathscr{H}$:
    A path γ is an $\mathscr{H}$-geodesic if and only if $\ddot{\gamma} = Q(\dot{\gamma})$;
  - the local Christoffel forms:
    A path $\gamma : I \to U_\alpha$ is a local $\mathscr{H}$-geodesic if and only if $\ddot{\gamma} = \Gamma^\alpha(\dot{\gamma}, \dot{\gamma})$.

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Totally Geodesic Submanifolds

- A submanifold $\iota : N \hookrightarrow M$ is *totally geodesic* if for every geodesic $\gamma \subset N$, $\iota(\gamma)$ " $=$ " γ is a geodesic in M.
- Geodesics satisfy $\ddot{\gamma} = Q(\gamma)$, and Q is preserved under the inclusion map ι: $\iota_{**}Q = Q(\iota_*)$, but $\iota_* : TN \hookrightarrow TM$ is the inclusion of the tangent spaces.
- Therefore, a submanifold $\iota : N \twoheadrightarrow M$ is totally geodesic if and only if $Q(TN) = TTN \leq \mathscr{H}$.
- This makes perfect sense, as totally geodesic submanifolds are those which appear flat to observers in M.

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Quotation

from *A Sand County Almanac*

To those devoid of imagination a blank place on the map is a useless waste; to others, the most valuable part.

– Aldo Leopold

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Thank you!