26 Feb 2014

Surgery and Cobordism.

A *surgery* on a smooth mfld X^n is the construction of a new n-mfld X' by removing an embedded p-sphere from X and replacing it w/ a q-sphere where $p+q+1=n$.

Spse

$$i : S^p \hookrightarrow X$$

is an embedding, with trivial normal bundle.
Then we can extend i to an embedding $\bar{i} : S^p \times D^{q+1} \hookrightarrow X$.
$\bar{i}$ is a *framed embedding* of S^p.

By removing an open nbd of S^p, we obtain a mfld

$X \setminus \bar{i}(S^p \times \mathring{D}^{q+1})$ w/ boundary $S^p \times S^q$.

As the handle $D^{p+1} \times S^q$ has the same boundary, we can use $\bar{i}|_{S^p \times S^q}$ to glue the manifolds $X \setminus \bar{i}(S^p \times \mathring{D}^{q+1})$ and $D^{p+1} \times S^q$ along their common boundary to obtain a new manifold

$$X' = (X \setminus \bar{i}(S^p \times \mathring{D}^{q+1})) \cup_{\bar{i}} (D^{p+1} \times S^q).$$

We may assume that the attaching was smooth. (In reality, it needs to be smoothed using cylindrical "cuffs".)

We all know what a cobordism is, but let's recall.

A <u>cobordism</u> between n-mflds X_0 and X_1 is an $(n+1)$-dim mfld $W^{n+1} = \{W^{n+1}; X_0, X_1\}$ w/ boundary $X_0 \sqcup X_1 = \partial W$.

Cobordisms which arise as traces of a surgery are known as <u>elementary cobordisms</u>.

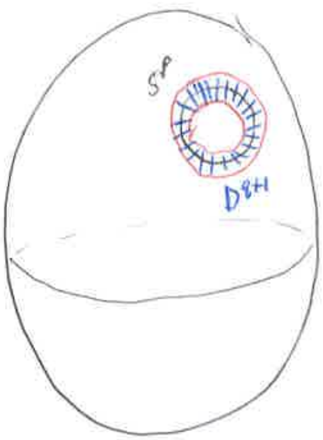
An important consequence of Morse theory:

Any cpt cobordism $\{W^{n+1}; X_0, X_1\}$ may be decomposed as a finite union of elementary cobordisms.

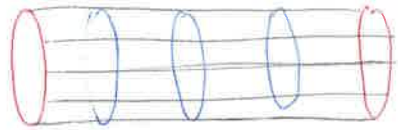
<u>The Surgery Theorem</u> (Gromov-Lawson; Schoen-Yau):

(X, g) a Riemannian mfld of positive scalar curvature. Let X' be a mfld obtained from X by a surgery of codim at least 3. Then X' admits a psc metric g'.

M^n



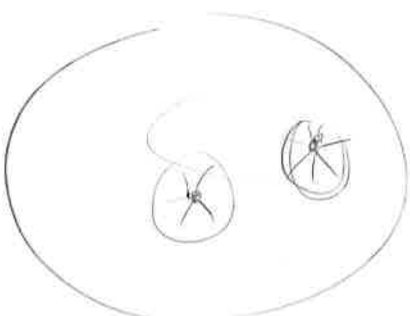
$\partial(S^p \times D^{q+1})$

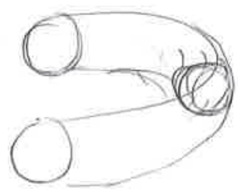


$D^{p+1} \times S^q$

$\partial(D^{p+1} \times S^q) = \partial(S^p \times D^{q+1})$

Glue this "handle" to M along the common boundary.





— $S^{q} \times D^{p+1}$

$\mathrm{Diff}(S^p \times S^q)$

$S^p \times D^{q+1}$