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S^n, $n \geq 3$, standard n-sphere.

We shall consider warped and doubly warped product metrics on S^n. All will have non-negative sectional and Ricci curvature, and positive scalar curvature.

Denote the standard round metric of radius 1 by ds_n^2.

Ex. Consider the embedding

$$(0,\pi) \times S^{n-1} \longrightarrow \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$$

$$(t, \theta) \longmapsto (\cos t, \sin t \cdot \theta)$$

This embedding gives rise to the metric

$dt^2 + \sin^2(t)\, ds_{n-1}^2$ on $(0,\pi) \times S^{n-1}$, and extends uniquely to the round metric of radius 1 on S^n.

Similarly, the round metric of radius ε takes the form $dt^2 + \varepsilon^2 \sin^2\left(\frac{t}{\varepsilon}\right) ds_{n-1}^2$ on $(0, \varepsilon\pi) \times S^{n-1}$.

More generally, replacing $\sin t$ w/ a suitable smooth function $f : (0,b) \to (0,\infty)$, we can construct other metrics on S^n.

We need criteria for such a metric to be smooth on S^n.

Prop. $f : (0,b) \to (0,\infty)$ smooth w/ $f(0) = f(b) = 0$.

The metric $g = dt^2 + f(t)^2 \, ds^2_{n-1}$ is a smooth metric on S^n iff $f^{(\text{even})}(0) = 0$, $\dot f(0) = 1$, and $f^{(\text{even})}(b) = 0$, and $\dot f(b) = -1$.

A metric of this form on S^n is known as a <u>warped product metric</u>.

The Ricci and scalar curvatures of such a metric are given by the following. Let $\partial_t, e_1, \dots, e_{n-1}$ be an orthonormal frame.

Then,

$$\boxed{\begin{aligned} Ric(\partial_t) &= -(n-1)\frac{\ddot f}{f} \\ Ric(e_i) &= (n-2)\frac{1-\dot f^2}{f^2} - \frac{\ddot f}{f} \end{aligned}} \qquad i = 1, \dots, n-1$$

And the scalar curvature is thus:

$$\boxed{S = -2(n-1)\frac{\ddot f}{f} + (n-1)(n-2)\frac{1-\dot f^2}{f^2}}$$

Let $\mathcal{F}(0,b)$ denote the space of smooth functions satisfying

$$f(0) = 0 \qquad f(b) = 0$$

$$\dot f(0) = 1 \qquad \dot f(b) = -1$$

$$f^{(\text{even})}(0) = f^{(\text{even})}(b) = 0$$

$$\ddot f \le 0, \quad \dddot f(0) < 0, \quad \dddot f(b) > 0$$

$$\ddot f(t) < 0 \quad \text{for } t \text{ near but not at } 0, b.$$

Typical elements look like figure on p. 25.

<u>Prop.</u> The space $W(0,b) = \{ dt^2 + f(t)^2 ds^2_{n-1} \mid f \in \mathcal{F}(0,b) \}$ is a path connected subspace of $Riem^+(S^n)$.

Let $\mathcal{F} = \bigcup_{b \in (0,\infty)} \mathcal{F}(0,b)$, and $\mathcal{W} = \bigcup_{b \in (0,\infty)} \mathcal{W}(0,b)$.

<u>Prop.</u> The space $\mathcal{W}$ is a path connected subspace of $Riem^+(S^n)$.