Following: Walsh, "Metrics of psc and Morse Functions" 12 Feb 14

Let X be a smooth closed cpt n-mfld. later: $n \geq 5$, X simply conn.

$Riem(X)$ = space of Riemannian metrics on X.

The topology on this space is induced by the C^k-norm on Riemannian metrics:

Let g be any metric in $Riem(X)$, [Recall that $Riem(X)$ is nonempty for all X.] and ∇ the LC-connection for g.

Let $\|\nabla^i g\|$ be the standard Euclidean tensor norm on ∇g.

Define $\|g\|_k = \max_{i \leq k} \sup_X \|\nabla^i g\|$.

The topology on $Riem(X)$ does not depend on choice of g.

We shall (usually) assume that $k = 2$.

[Notice that ∇^2 is "like a Laplacian".]

Let $Riem^+(X) = \{ g \in Riem(X) \mid S_g > 0 \}$

be the space of psc metrics on X. $Riem^+(X)$ is open in $Riem(X)$.

whether or not a given X admits a psc metric has been studied extensively. [We'll return to this later in the seminar.]

Thus, we shall assume $\mathcal{R}iem^+(X) \neq \emptyset$.

<u>Ex.</u> $\mathcal{R}iem^+(X)$ is a convex space.

i.e., $g_0, g_1 \in \mathcal{R}iem^+(X) \Rightarrow s g_0 + (1-s) g_1 \in \mathcal{R}iem^+(X) \quad \forall s \in [0,1]$.

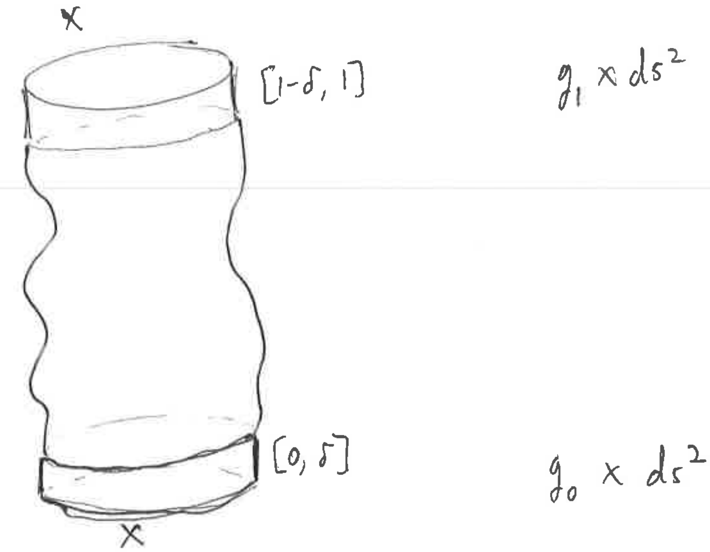
The topology of $\mathcal{R}iem^+(X)$ is <u>not</u> well-understood.

We shall need two equivalence relations on X:

<u>Defn.</u> Two metrics g_0 and g_1 are said to be <u>isotopic</u> if they lie in the same path component of $\mathcal{R}iem^+(X)$.
A path g_s, $s \in I$, w/ $g_s(0) = g_0$ and $g_s(1) = g_1$ is said to be an <u>isotopy</u>.

<u>Defn.</u> If $\exists$ a psc metric $\bar{g}$ on the cylinder $X \times I$ s.t. for some $\delta > 0$, $\bar{g}|_{X \times [0,\delta]} = g_0 + ds^2$ and $\bar{g}|_{X \times [1-\delta, 1]} = g_1 \times ds^2$, then g_0 and g_1 are said to be <u>concordant</u>. The metric $\bar{g}$ is known as a <u>concordance</u>.

Concordance:



Lemma. Let g_r be a smooth path in $Riem^+(X)$, $r \in I$. $\exists$ a constant $0 < \Lambda \leq 1$ s.t. $\forall$ smooth $f: \mathbb{R} \to [0,1]$ w/ $\|\dot{f}\|, \|\ddot{f}\| \leq \Lambda$, the metric $g_{f(s)} + ds^2$ on $X \times \mathbb{R}$ has psc.

Corollary. Metrics which are isotopic are also concordant.

Question: When are concordant metrics isotopic?

One difficulty in answering this question is that in general a concordance can be extremely complicated.

viz. p.7 of Mark's thesis.

Thus, we will not be able to answer this question in full generality

Consequently, we will not approach this problem at the level of arbitrary concordance. Instead, we will restrict our attention to concordances which are constructed by a particular application of the surgery technique of Gromov and Lawson. Such concordances will be called *Gromov-Lawson concordances.* Before discussing the relationship between surgery and concordance, it is worth recalling how the surgery technique alters a psc-metric.

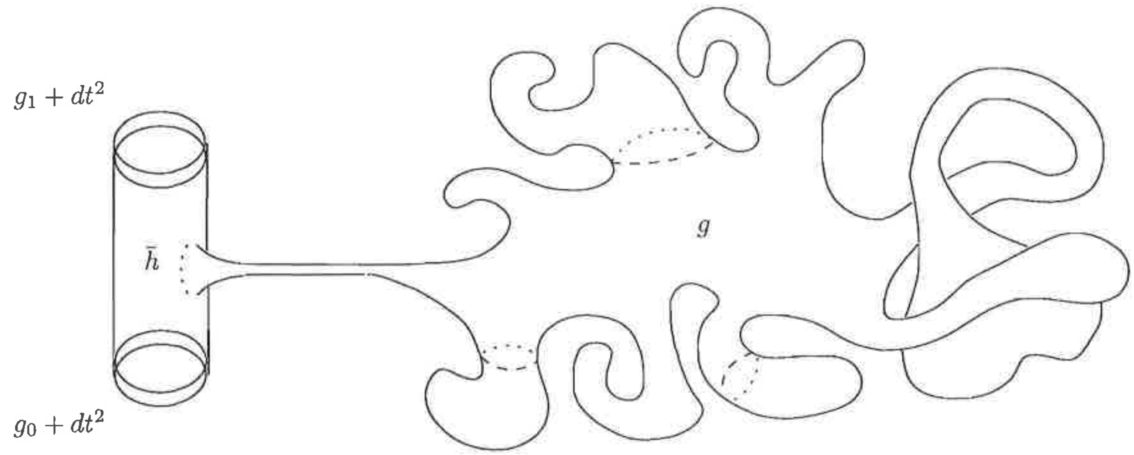


Figure I.2: The concordance $\bar{g}$ on $S^n \times I$, formed by taking a conncected sum of metrics $\bar{h}$ and g.

I.2.3 Surgery and positive scalar curvature

We begin by stating the Surgery Theorem of Gromov-Lawson and Schoen-Yau.

Surgery Theorem.([14], [38]) *Suppose X admits a psc-metric and X' is a manifold which is obtained from X by surgery in codimension ≥ 3. Then X' admits a psc-metric also.*

In their proof, Gromov and Lawson show that a psc-metric g on X can be replaced with a psc-metric g_{std} which is standard in a tubular neighbourhood of the embedded surgery sphere. More precisely, let ds_n^2 denote the standard round metric on the sphere S^n. We denote by $g_{tor}^n(\delta)$, the metric on the disk D^n which, near ∂D^n, is the Riemannian cylinder $\delta^2 ds_{n-1}^2 + dr^2$ and which near the centre of D^n is the round metric $\delta^2 ds_n^2$. The metric $g_{tor}^n(\delta)$ is known as a *torpedo metric,* see section II.2 for a detailed construction. For sufficiently small $\delta > 0$ and provided $n \geq 3$, the scalar curvature of this metric can be bounded below by an arbitrarily large positive constant. Now, let (X, g) be a smooth n-dimensional Riemannian manifold of positive scalar curvature and let S^p denote an embedded p-sphere in X with trivial normal bundle and with $p + q + 1 = n$ and $q \geq 2$.