M: smooth mfld

TM: tangent bundle

T^*M: cotangent bundle

O'Neill

A Riemannian metric tensor is an assignment of positive definite inner product (symm. non-deg. bilinear form) to each tangent space T_pM that varies smoothly between fibers of TM.

Thm. Every smooth mfld M admits a Riemannian metric.

Pf. Put the standard inner product on a single fiber, then "spread it out" over the mfld using a partition of unity.

PoU: $\{U_i\}$ open cover of M. Since M is paracompact, $\exists$ a partition $\{\rho_i\}$ s.t. supp $\rho_i \subseteq U_i$. We say $\{\rho_i\}$ is subbordinate to $\{U_i\}$.
(Existence ?)

Defn. A **PoU** of a top. space X is a set of continuous functions from X to $I = [0,1]$ s.t. for all $x \in X$,

1. There is a nbd of x where all but finitely many functions are 0, and
2. The sum of all of the values at x is 1.

$$\sum_{\rho \in R} \rho(x) = 1, \quad \forall x \in X.$$

Alternatively, a Riemannian metric tensor is a section of the bundle $T^*M \otimes T^*M$,

$$g \in \Gamma(T^*M \otimes T^*M).$$

Once we have a metric:

T_pM has dual space T_p^*M.

There is an isomorphism induced by the metric:

$$X_p \in T_pM \cong \langle X_p, \ \rangle_p \in T_p^*M \quad \text{and}$$

$$X \in \mathfrak{X} \cong g(X, \ ) \in \mathfrak{X}^* = \Omega^1$$

We say that $\langle X, \ \rangle =: X^\flat$ is the one-form <u>metrically equivalent</u> to X.

Similarly, for $\theta \in \Omega^1$, $\theta^\sharp$ is the vector field m.e. to θ.

To talk about curvature, we need a connection.

<u>Defn.</u> A <u>connection</u> is on M is a function

$$\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X} \quad \text{s.t.}$$

1. $\nabla_V W$ is $\mathfrak{F}$-linear in V
2. $\nabla_V W$ is $\mathbb{R}$-linear in W
3. $\nabla_V(fW) = (Vf)W + f\nabla_V W$, $f \in \mathfrak{F}$.

For a Riemannian mfld (M, g), there is a unique connection that also satisfies:

4. $[V, W] = \nabla_V W - \nabla_W V$ and
5. $X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$

This ∇ is called the Levi-Civita connection for g, and satisfies the Koszul formula:

$$2\langle \nabla_V W, X\rangle = V\langle W, X\rangle + W\langle X, V\rangle - X\langle V, W\rangle$$
$$- \langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle$$

The Riemann Curvature Tensor on (M, g) is the function

$R: \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ given by:

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$
$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

* Notice that ∇ is not a tensor, but this particular combination of ∇'s is a tensor.

tensor $\cong$ "defined at a point"

Prop. R has considerable symmetry: Let $x, y, z, v, w \in T_pM$, then

1. $R(X,Y) = -R(Y,X)$
2. $\langle R(X,Y)V, W\rangle = -\langle R(X,Y)W, V\rangle$
3. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ First Bianchi Id.
4. $\langle R(X,Y)V, W\rangle = \langle R(V,W)X, Y\rangle$

Since $R: \mathfrak{X}^3 \to \mathfrak{X}$, ∇R makes sense, and can be thought of as a $(1,4)$-tensor, $\nabla R: \mathfrak{X}^4 \to \mathfrak{X}$.
We obtain the second Bianchi Id by combining the 4 above:

$x, y, z \in T_pM$,

$$(\nabla_z R)(x,y) + (\nabla_x R)(y,z) + (\nabla_y R)(z,x) = 0$$

Sectional Curvature

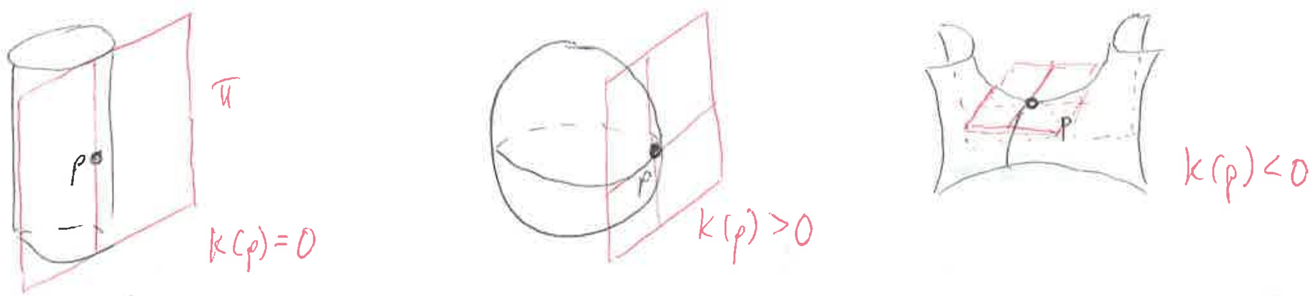
Let $v, w \in T_pM$ be linearly independent, and let $\pi = \pi(v,w)$ be the tangent plane spanned by v, w.

Define $Q(v,w) = \langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2 \neq 0$.

Lemma. The number $K(\pi) = \dfrac{\langle R(v,w)v, w\rangle}{Q(v,w)}$ is independent of the choice of basis v, w for π.
$K(\pi)$ is called the sectional curvature of π.

For a 2-dim mfld M (a surface), then K is a function on M. For $\dim M > 2$, K depends on the plane π.

Geometrically, for a surface:



The other curvatures are made by "averaging" or contracting R.

Metric Contraction:

Let A be an (r,s)-tensor; i.e., A takes r 1-forms and s v.f.s and gives a number.

Write $A \in \mathcal{T}^r_s(M)$.

$$A(\theta^1, \theta^2, \dots, \theta^r, X_1, \dots, X_s) \in \mathbb{R}$$

A can be made into a $\mathcal{T}^{r-1}_{s+1}$ ~~via the metric~~ OK

$$(\overset{a}{\underset{b}{\downarrow}} A)(\theta^1, \dots, \theta^{r-1}, X_1, \dots, X_{s+1}) := A(\underbrace{\theta^1, \dots, X_b^{\flat}, \dots, \theta^{r-1}}_{r-1+1=r}, \underbrace{X_1, \dots, X_{b-1}, X_{b+1}, \dots, X_{s+1}}_{s+1-1=s})$$

a^{th} slot

This is known as lowering an index. One can analogously define "raising an index" by sending $\theta_a \mapsto \theta_a^{\sharp}$ in the b^{th} slot.

Not Correct!

We can also contract a tensor by applying a 1-form to a v.f.

$(C^a_b A) \in \mathcal{T}^{r-1}_{s-1}$

$$C^a_b A = \underbrace{(\theta^a X_b)}_{\in \mathfrak{F}} A(\underbrace{\theta^1,\dots,\theta^{a-1},\theta^{a+1},\dots,\theta^r}_{r-1}, \underbrace{X_1,\dots,X_{b-1},X_{b+1},\dots,X_s}_{s-1})$$

Metric contraction combines contraction w/ raising or lowering indices.

e.g., $(C_{ab} A) \in \mathcal{T}^r_{s-2}$

$$(C_{ab} A) = \underbrace{(X_a^b X_b)}_{\in \mathfrak{F}} A(\theta^1,\dots,\underbrace{\theta^r}_{r}, \underbrace{X_1,\dots,X_{a-1},X_{a+1},\dots,X_{b-1},X_{b+1}\dots X_s}_{s-2})$$

Now recall that $R \in \mathcal{T}^1_3$

Defn. The Ricci Curvature tensor is the contraction

$$Ric = (C^1_3 R) \in \mathcal{T}^0_2$$

If $\{E_i\}$ is a frame field on M, then Ric is given by:

$$Ric(X,Y) = \sum_m \langle R(X,E_m)Y, E_m \rangle$$

So $Ric(X,Y)$ is like an average of sectional curvatures.

Defn. The scalar curvature S of (M,g) is the metric contraction $C(Ric) \in \mathfrak{F}(M)$.

$$S = \sum_{i \neq j} K(E_i,E_j) = 2\sum_{i<j} K(E_i,E_j)$$

So for a surface

$S(p) = 2K(p)$, and there is really only one curvature.

Geometrically, the scalar curvature at a point represents the amount by which the volume of a geodesic ball around the point deviates from that of a Euclidean ball.

positive = smaller volume = geodesics come back together
negative = larger volume = geodesics spread apart.