Subsequences of the Fibonacci Sequence

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1 The Fibonacci and Lucas Numbers

In this note, we will develop a collection of sequences each of which is a subsequence of the Fibonacci sequence. Each of these sequences has the property that the quotient of consecutive terms converges to a power of the golden ratio. We will begin with a review of the Fibonacci sequence and some of its properties as well as examine a the sequence that yields the Lucas numbers. The sequences we will define involve Fibonacci and Lucas numbers in their definitions.

The n^{th} term of the Fibonacci sequence will be denoted by F_n. The sequence itself is given by

$$F_{n+1} = F_n + F_{n-1}, \text{ with } F_0 = F_1 = 1.$$

The first few terms of the Fibonacci sequence are then

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, \ldots.$$

If we compute the quotients of consecutive terms, we get

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \frac{233}{144}, \frac{377}{233}, \frac{610}{377}, \frac{987}{610}, \frac{1597}{987}, \frac{2584}{1597}, \frac{4181}{2584}, \ldots$$

The decimal representation of these quotients appear to hover around the value 1.61. The quotient $\frac{4181}{2584} = 1.618034056$.

As it turns out, among the many results that have been discovered about the Fibonacci sequence over the years is that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} = \varphi, \quad \text{the golden ratio.}$$

This limit can be obtained using the result called the Binet formula [1]

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right], \quad n = 0, 1, 2, \ldots.$$

If we let $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \dfrac{1-\sqrt{5}}{2}$, we can then write

$$F_n = \frac{1}{\sqrt{5}} \left[\varphi^{n+1} - \overline{\varphi}^{n+1} \right], \quad n = 0, 1, 2, \ldots.$$

The French mathematician Franois Édouard Anatole Lucas (1842–1891) made a detailed study of the Fibonacci sequence and related sequences. A sequence of the form $f_{n+1} = f_n + f_{n-1}$ where f_0 and f_1 are given is referred to as a Fibonacci-type sequence. Lucas came up with his own Fibonacci-type sequence given by

$$L_{n+1} = L_n + L_{n-1}, \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

This yields the sequence of numbers

$$2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots.$$

One might reasonably ask why start with 2, 1 instead of 1, 3. If the sequence starts with 2, 1, then the de Moivre-Binet formula is very nice

$$L_n = \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n = \varphi^n + \overline{\varphi}^n, \quad n = 0, 1, 2, \ldots.$$

In addition one can define $L_{-k} = (-1)^k L_k$, for $k = 1, 2, 3, \ldots$ if the sequence L_k starts 2, 1.

[1] There appears to be some controversy as to the credit given for this formula. Most refer to it as the Binet formula in honor of J. P. M. Binet (1786-1856) a French mathematician who published the formula in 1843. However, some say Abraham de Moivre (1667-1754) obtained the result one hundred years earlier in 1730, so we will call it the de Moivre-Binet formula.

2 Subsequences of the Fibonacci Sequence

We are going to use the Fibonacci numbers and the Lucas numbers to define a collection of sequences as follows

$$x_{k,n} = L_k x_{k,n-1} + (-1)^{k-1} x_{k,n-2} \text{ with } x_{k,0} = F_0 \text{ and } x_{k,1} = F_k,\ k \geq 1, n \geq 2.$$

The sequence

$$x_{1,n} = L_1 x_{1,n-1} + (-1)^0 x_{1,n-1} = x_{1,n-1} + x_{1,n-2} \text{ with } x_{1,0} = 1,\ x_{1,1} = 1,$$

is merely the Fibonacci sequence. The following table gives nineteen terms for the first four of our sequences.

n	$x_{1,n}$	$x_{2,n}$	$x_{3,n}$	$x_{4,n}$
0	1	1	1	1
1	1	2	3	5
2	2	5	13	34
3	3	13	55	233
4	5	34	233	1597
5	8	89	987	10946
6	13	233	4181	75025
7	21	610	17711	514229
8	34	1597	75025	3524578
9	55	4181	317811	24157817
10	89	10946	1346269	165580141
11	144	28657	5702887	1134903170
12	233	75025	24157817	7778742049
13	377	196418	102334155	53316291173
14	610	514229	433494437	365435296162
15	987	1346269	1836311903	2504730781961
16	1597	3524578	7778742049	17167680177565
17	2584	9227465	32951280099	117669030460994
18	4181	24157817	139583862445	806515533049393

In this table, the second column is the Fibonacci sequence, the third column appears to be every other term of the Fibonacci sequence, the fourth column appears to be every third term of the Fibonacci sequence and the fifth column appears to have every fourth term of the Fibonacci sequence. Moreover, $\dfrac{x_{2,18}}{x_{2,17}}$ is very close to the square of the golden ratio, $\dfrac{x_{3,18}}{x_{3,17}}$ is very close to the cube of the golden ratio, and $\dfrac{x_{4,18}}{x_{4,17}}$ is very close to the fourth power of the golden ratio.

We will establish that what we see from the table is in fact true. That is, $x_{k,n}$ generates every k^{th} term of the Fibonacci sequence and that as n goes to ∞

$$\frac{x_{k,n+1}}{x_{k,n}} \to \varphi^k \text{ for } k = 1, 2, 3, \ldots.$$

We begin with some lemmas.

Lemma 2.1. $L_k^2 + (-1)^{k-1}4 = 5F_{k-1}^2$.

Proof. In the following, we use the fact that $\varphi\overline{\varphi} = -1$.

$$\begin{aligned}
L_k^2 + (-1)^{k-1} &= (\varphi^k + \overline{\varphi}^k)^2 + (-1)^{k-1}4 \\
&= \varphi^{2k} + 2\varphi^k\overline{\varphi}^k + \overline{\varphi}^{2k} + (-1)^{k-1}4 \\
&= \varphi^{2k} + 2(-1)^k + \overline{\varphi}^{2k} + (-1)^{k-1}4 \\
&= \varphi^{2k} - 2(-1)^k + \overline{\varphi}^{2k} \\
&= (\varphi^k - \overline{\varphi}^k)^2 = 5F_{k-1}^2
\end{aligned}$$

Lemma 2.2. $L_k + \sqrt{5}F_{k-1} = 2\varphi^k$ *and* $L_k - \sqrt{5}F_{k-1} = 2\overline{\varphi}^k$

Proof. This is just a matter of writing out what the terms are.

$$\begin{aligned}
L_k + \sqrt{5}F_{k-1} &= \varphi^k + \overline{\varphi}^k + \varphi^k - \overline{\varphi}^k = 2\varphi^k \\
L_k - \sqrt{5}F_{k-1} &= \varphi^k + \overline{\varphi}^k - \varphi^k + \overline{\varphi}^k = 2\overline{\varphi}^k
\end{aligned}$$

Lemma 2.3. $\dfrac{F_k - \overline{\varphi}^k}{F_{k-1}} = \varphi$.

Proof. We begin by calculating $\varphi F_{k-1} + \overline{\varphi}^k$. Again we will use the fact that $\varphi\overline{\varphi} = -1$.

$$
\begin{aligned}
\varphi F_{k-1} + \overline{\varphi}^k &= \varphi \left[\frac{1}{\sqrt{5}} (\varphi^k - \overline{\varphi}^k) \right] + \overline{\varphi}^k \\
&= \frac{1}{\sqrt{5}} \varphi^{k+1} - \frac{1}{\sqrt{5}} \varphi \overline{\varphi}^k + \overline{\varphi}^k \\
&= \frac{1}{\sqrt{5}} \varphi^{k+1} + \frac{1}{\sqrt{5}} \overline{\varphi}^{k-1} + \overline{\varphi}^k \\
&= \frac{1}{\sqrt{5}} \varphi^{k+1} + \frac{1}{\sqrt{5}} \overline{\varphi}^{k-1} (1 + \sqrt{5}\overline{\varphi}) \\
&= \frac{1}{\sqrt{5}} \varphi^{k+1} + \frac{1}{\sqrt{5}} \overline{\varphi}^{k-1} (-\overline{\varphi}^2) \\
&= \frac{1}{\sqrt{5}} (\varphi^{k+1} - \overline{\varphi}^{k+1}) \\
&= F_k.
\end{aligned}
$$

Therefore, solving for φ we have $\dfrac{F_k - \overline{\varphi}^k}{F_{k-1}} = \varphi$.

We are now ready to prove the major theorem of this note.

Theorem 2.1. $x_{k,n} = F_{nk}$.

Proof. For the recurrence relation $x_{k,n} = L_k x_{k,n-1} + (-1)^{k-1} x_{k,n-2}$ with $x_{k,0} = 1,\ x_{k,1} + F_k$, we have the auxiliary equation

$$r^2 - L_k r - (-1)^{k-1} = 0$$

and

$$r = \frac{L_k \pm \sqrt{L_k^2 + (-1)^{k-1} 4}}{2}.$$

From Lemma 2.1, this becomes $r = \dfrac{L_k \pm \sqrt{5} F_{k-1}}{2}$. Then an application of Lemma 2.2, yields

$$r_1 = \varphi^k \ \text{ and } \ r_2 = \overline{\varphi}^k.$$

Hence

$$x_{k,n} = A(\varphi^k)^n + B(\overline{\varphi}^k)^n \ \text{ with } x_{k,0} = 1,\ x_{k,1} = F_k.$$

Using the initial conditions that $x_{k,0} = 1$ and $x_{k,1} = F_k$ we obtain,

$$A + B = 1 \tag{1}$$

$$\varphi^k A + \overline{\varphi}^k B = F_k \tag{2}$$

Multiplying equation (1) by $\overline{\varphi}^k$ and subtracting from equation (2) gives

$$\begin{aligned}(\varphi^k - \overline{\varphi}^k)A &= F_k - \overline{\varphi}^K \\ A &= \frac{F_k - \overline{\varphi}^k}{\varphi^k - \overline{\varphi}^k} = \frac{F_k - \overline{\varphi}^k}{\sqrt{5}F_{k-1}}\end{aligned}$$

An application of Lemma 2.3 reduces this to $A = \frac{1}{\sqrt{5}}\varphi$ and then $B = -\frac{1}{\sqrt{5}}\overline{\varphi}$. Thus,

$$\begin{aligned}x_{k,n} &= \frac{1}{\sqrt{5}}\varphi(\varphi^k)^n - \frac{1}{\sqrt{5}}\overline{\varphi}(\overline{\varphi}^k)^n \\ &= \frac{1}{\sqrt{5}}(\varphi^{nk+1} - \overline{\varphi}^{nk+1}) \\ &= F_{nk}.\end{aligned}$$

Theorem 2.2. $\lim_{n\to\infty} \frac{x_{k,n+1}}{x_{k,n}} = \varphi^k$.

Proof. We note that since $\varphi > 1$ and $\overline{\varphi} < 1$, it follows that $\frac{\overline{\varphi}}{\varphi} < 1$.

$$\begin{aligned}\frac{x_{k,n+1}}{x_{k,n}} = \frac{F_{(n+1)k}}{F_{nk}} &= \frac{\varphi^{(n+1)k+1} - \overline{\varphi}^{(n+1)k+1}}{\varphi^{nk+1} - \overline{\varphi}^{nk+1}} \\ &= \frac{\frac{\varphi^{(n+1)k+1}}{\varphi^{nk+1}} - \frac{\overline{\varphi}^{(n+1)k+1}}{\varphi^{nk+1}}}{1 - \left(\frac{\overline{\varphi}}{\varphi}\right)^{nk+1}} \\ &= \frac{\varphi^k - \overline{\varphi}\left(\frac{\overline{\varphi}}{\varphi}\right)^{nk+1}}{1 - \left(\frac{\overline{\varphi}}{\varphi}\right)^{nk+1}}\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{x_{k,n+1}}{x_{k,n}} = \lim_{n\to\infty} \frac{\varphi^k - \overline{\varphi}\left(\frac{\overline{\varphi}}{\varphi}\right)^{nk+1}}{1 - \left(\frac{\overline{\varphi}}{\varphi}\right)^{nk+1}} = \varphi^k$$

3 An Interesting Identity

In searching for a closed form formula for $x_{k,n}$ two different methods of solving the recurrence relation were used and one ended up with a more complex closed form. However, putting the two solutions together has resulted in an interesting identity. The following theorem gives an alternate method of solving a recurrence relation.

Theorem 3.1. *Let $x_{n+1} = ax_n + bx_{n-1}$ with x_0 and x_1 given. If r_1 and r_2 are distinct roots [2] of $r^2 - ar - b = 0$, then*

$$x_n = \frac{(x_1 - r_1 x_0) r_2^n - (x_1 - r_2 x_0) r_1^n}{r_2 - r_1} \tag{3}$$

Proof. If r_1 and r_2 are the distinct roots of $r^2 - ar - b = 0$, then $a = r_1 + r_2$ and $b = -r_1 r_2$. We can then write

$$x_{n+1} - ax_n = bx_{n-1}$$

as

$$x_{n+1} - (r_1 + r_2)x_n = -r_1 r_2 x_{n-1}.$$

Thus

$$x_{n+1} - r_1 x_n = r_2(x_n - r_1 x_{n-1}) \tag{4}$$

and

$$x_{n+1} - r_2 x_n = r_1(x_n - r_2 x_{n-1}) \tag{5}$$

Applying the recurrence relation (4) to itself yields

$$x_{n+1} - r_1 x_n = r_2^n (x_1 - r_1 x_0) \tag{6}$$

In a similar manner (5) gives

$$x_{n+1} - r_2 x_n = r_1^n (x_1 - r_2 x_0) \tag{7}$$

Now subtract (6) from (7) to obtain

$$(r_1 - r_2)x_n = r_1^n (x_1 - r_2 x_0) - r_2^n (x_1 - r_1 x_0)$$

[2] We only need the result for distinct roots.

which, in turn, yields

$$x_n = \frac{(x_1 - r_2 x_0) r_1^n - (x_1 - r_1 x_0) r_2^n}{r_1 - r_2}.$$

If we apply this lemma to

$$x_{k,n} = L_k x_{k,n-1} + (-1)^{k-1} x_{k,n-2} \quad \text{with } x_{k,0} = 1,\ x_{k,1} = F_k,$$

we have, recalling that $r_1 = \varphi^k$ and $r_2 = \overline{\varphi}^k$,

$$\begin{aligned}
x_{k,n} &= \frac{(F_k - \overline{\varphi}^k)\varphi^{nk} - (F_k - \varphi^k)\overline{\varphi}^{nk}}{\varphi^k - \overline{\varphi}^k} \\
&= \frac{F_k \varphi^{nk} - \overline{\varphi}^k \varphi^{nk} - F_k \overline{\varphi}^{nk} + \varphi^k \overline{\varphi}^{nk}}{\sqrt{5} F_{k-1}} \\
&= \frac{F_k(\varphi^{nk} - \overline{\varphi}^{nk}) - \varphi^k \overline{\varphi}^k (\varphi^{(n-1)k} - \overline{\varphi}^{(n-1)k})}{\sqrt{5} F_{k-1}} \\
&= \frac{\sqrt{5} F_k F_{nk-1} - (-1)^k \sqrt{5} F_{(n-1)k-1}}{\sqrt{5} F_{k-1}} \\
&= \frac{F_k F_{nk-1} + (-1)^{k-1} F_{(n-1)k-1}}{F_{k-1}}
\end{aligned}$$

However in Theorem 2.1 we found $x_{k,n} = F_{nk}$. Therefore equating the two forms for $x_{k,n}$, we have

$$F_{nk} F_{k-1} = F_k F_{nk-1} + (-1)^{k-1} F_{(n-1)k-1} \quad \text{with } n \geq 2,\ k \geq 1.$$