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Strictly positive definite functions on spheres

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Abstract

In this paper, we proved that a positive definite radial function on $\mathbb{R}^d$ with support in $[0, \pi]$ is strictly positive definite on the sphere $\mathbb{S}^d$ and real projective space $\mathbb{RP}^d$ for odd $d \geq 3$. We also proved that the truncated power function $(t - \cdot)_+^{(d+1)/2}$ is strictly positive definite on $\mathbb{S}^d$ and $\mathbb{RP}^d$ for $d \geq 2$ and $t \in (0, \pi]$.

MSC: 60G60

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1. Introduction

Positive definite (PD) functions and strictly positive definite (SPD) functions play important roles in the approximation theory. A continuous (radial) function $f : [0, \infty) \to \mathbb{R}$ is PD on $\mathbb{R}^d$ if for all distinct point sets $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ in $\mathbb{R}^d$, the matrix $[f(\rho(\mathbf{x}_i, \mathbf{x}_j))]_{i,j=1}^n$ is positive semidefinite, where $\rho(\mathbf{x}, \mathbf{y})$ denotes the Euclidean distance. The function f is SPD if such matrices are all positive definite. PD and SPD functions on the sphere $\mathbb{S}^d = \{\mathbf{x} \in \mathbb{R}^{d+1}, \|\mathbf{x}\| = 1\}$ or the real projective space $\mathbb{RP}^d$ (formed by identifying antipodal points of $\mathbb{S}^d$) are defined accordingly for point sets in $\mathbb{S}^d$ or $\mathbb{RP}^d$ with $\rho(\mathbf{x}, \mathbf{y})$ denoting the geodesic distance.

Bochner's theorem characterizes PD functions on $\mathbb{R}^d$ by the nonnegativeness of the Fourier transform of the function $f(\|\cdot\|)$, i.e., $\int_{\mathbb{R}^d} f(\|\mathbf{x}\|)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \geq 0$. PD functions on the spheres have wide applications in geophysics and cosmology [12]. By [18], f is a PD function on $\mathbb{S}^d$ if and only if the spectral coefficients given below are nonnegative for $l \in \mathbb{N}_0$,

$$c_l(f) = \int_{\mathbb{S}^d} f(\rho(\mathbf{x}, \mathbf{y}))p_{\alpha,l}(\mathbf{x} \cdot \mathbf{y})d\mathbf{x}, \tag{1.1}$$

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where $p_{\alpha,l}(\cdot)$ is the normalized Jacobi polynomial (18.3 in [15]), $P_l^{(\alpha,\alpha)}(\cdot)/P_l^{(\alpha,\alpha)}(1)$, with $\alpha = (d-2)/2$. The normalized Jacobi polynomial $p_{\alpha,l}(\cdot)$ is the same as the normalized Gegenbauer polynomial (18.3 in [15]), $\hat{C}_l^{(\alpha)}(\cdot) = C_l^{(\alpha)}(\cdot)/C_l^{(\alpha)}(1)$, with $\alpha = (d-1)/2$.

SPD functions are particularly useful in the approximation theory because they guarantee the existence and uniqueness of the solution to the interpolation problem for $g(\cdot)$ [2],

$$\sum_{j=1}^{n} f(\rho(\mathbf{x}_i, \mathbf{x}_j))c_j = g(\mathbf{x}_i), \quad i = 1, \ldots, n.$$

We will study compactly supported SPD functions, for which the matrix in the interpolation problem is sparse and thus computationally efficient [20]. Compactly supported SPD functions on $\mathbb{R}^d$ have been studied for decades [19,20]. By the Paley–Wiener theorem [16], the Fourier transform of a function on $\mathbb{R}^d$ with compact support cannot vanish on an open set. As a result, by Theorem 2.2 in [5], a PD function on $\mathbb{R}^d$ with compact support is SPD on $\mathbb{R}^d$.

SPD functions on spheres and other two-point homogeneous spaces have been characterized in [1,6,9,13]. A PD function f on $\mathbb{S}^d$ with $d \geq 2$ is SPD if and only if its spectral coefficients $c_l(f) > 0$ for infinitely many even l and infinitely many odd l. A PD function f on $\mathbb{S}^1$ is SPD if and only if any infinite increasing integer arithmetic sequence contains an integer l such that $c_l(f) > 0$. A PD function f on a non-spherical two-point homogeneous space (e.g. a real projective space) if and only if its spectral coefficients $c_l(f) > 0$ for infinitely many l. In this paper, we will prove that certain functions have positive spectral coefficients on $\mathbb{S}^d$ or $\mathbb{RP}^d$, and therefore are SPD on the corresponding space.

One way to generate SPD functions on $\mathbb{S}^d$ is to restrict an SPD function on $\mathbb{R}^{d+1}$ to the embedded sphere $\mathbb{S}^d$ with Euclidean distance. If f is SPD on $\mathbb{R}^{d+1}$, $g(\cdot) = f(2\sin(\cdot/2))$ restricted to $[0, \pi]$ is SPD on $\mathbb{S}^d$. Another way to generate SPD functions on $\mathbb{S}^d$ is to use a compactly supported PD function on $\mathbb{R}^d$ directly for $\mathbb{S}^d$ with spherical distance. It has been proved that a PD function on $\mathbb{R}^d$ with support in $[0, \pi]$ is PD on $\mathbb{S}^d$ and real projective space $\mathbb{RP}^d$ for odd $d \geq 3$ [10,14]. We will prove that a PD function on $\mathbb{R}^d$ with support in $[0, \pi]$ has positive spectral coefficients on $\mathbb{S}^d$ and $\mathbb{RP}^d$ for odd $d \geq 3$, thus is SPD on those spaces. A PD function on $\mathbb{R}$ with support in $[0, \pi]$ is PD, but not necessarily SPD, on $\mathbb{S}^1$.

By our theorem, SPD functions on $\mathbb{S}^d$ for odd d can be generated from the compactly supported PD functions on $\mathbb{R}^d$ introduced in [19,20]. A natural question to ask is whether the theorem holds for even d. However, it remains an open problem. Compactly supported SPD functions on $\mathbb{S}^d$ and $\mathbb{R}^d$ for even d can be generated from such functions on $\mathbb{R}^{d+1}$. The literature on compactly supported PD functions on $\mathbb{R}^d$ for even d but not on $\mathbb{R}^{d+1}$ is scarce. There are a few ways to generate such functions. One way is to convolve a compactly supported function with itself on $\mathbb{R}^d$. Another way is to apply half-order integration to a compactly supported PD function $\phi(\cdot)$ on $\mathbb{R}^{d+1}$ as in [17],

$$f(x) = \int_x^{\infty} \phi(r)(r^2 - x^2)^{-1/2} r dr.$$

The third way is to use the truncated power function $(1-\cdot)_+^{\lambda}$ with $\lambda \geq (d+1)/2$, a classical PD function on $\mathbb{R}^d$ due to [8].

For even d, a PD function on $\mathbb{R}^d$ with support in $[0, \pi]$ is not guaranteed to be PD on $\mathbb{S}^d$. However, we will prove that it is guaranteed for a PD truncated power function. [2] proved that $(t-\cdot)_+^{(d+1)/2}$ was SPD on $\mathbb{S}^d$ for $d = 3, 5, 7$ and $t \in (0, \pi]$. By our result on odd dimensional spheres, the conclusion holds for all odd dimensions beyond 1. [7] proved that $(t-\cdot)_+^{3/2}$ is PD on $\mathbb{S}^2$ for sufficiently small t. We will prove Conjecture 1.4 in [2], which stated that for

$d \geq 2$, $t \in (0, \pi]$, and $\lambda \geq (d+1)/2$, the truncated power function $(t - \cdot)_+^\lambda$ has positive spectral coefficients on $\mathbb{S}^d$, and thus is SPD on $\mathbb{S}^d$.

In summary, we proved that a PD function on $\mathbb{R}^d$ with support in $[0, \pi]$ is SPD on $\mathbb{S}^d$ and $\mathbb{RP}^d$ for odd $d \geq 3$. As a result, the rich literature about compactly supported PD functions on $\mathbb{R}^d$ can be used to generate compactly supported SPD functions on $\mathbb{S}^d$, which has wide applications in geophysics. Although we did not prove the corresponding theorem for even d, we showed that it holds for two classes of compactly supported PD functions on $\mathbb{R}^d$, one of which being the truncated power functions. It sheds light on the conjecture that a PD function on $\mathbb{R}^d$ with support in $[0, \pi]$ is PD on $\mathbb{S}^d$ for all dimensions.

The rest of the paper is organized in the following way. In Section 2, we prove that a PD function on $\mathbb{R}^d$ with support in $[0, \pi]$ is SPD on $\mathbb{S}^d$ and $\mathbb{RP}^d$ for odd $d \geq 3$. In Section 3, we prove Conjecture 1.4 in [2] about truncated power functions on $\mathbb{S}^d$. In Section 4, we make some remarks and draw the conclusion.

2. Positive definite functions on odd dimensional spheres

The Fourier transform of a PD function f on $\mathbb{R}^d$, $\int_{\mathbb{R}^d} f(\|\mathbf{x}\|)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \geq 0$, is nonnegative and isotropic. We denote the radial part of the Fourier transform by $F(k)$ with $k = \|\mathbf{k}\|$. By the Paley–Wiener theorem, if f is compactly supported, the integral of F on any nonempty interval is positive. We will show that if $d \geq 3$ is odd, for $l \in \mathbb{N}_0$, the l'th spectral coefficient of f on $\mathbb{S}^d$, $c_l(f)$ defined by Eq. (1.1), can be written as

$$c_l(f) = \int_l^{l+d-1} F(k)B(k)dk,$$

for a continuous function B that is positive on $(l, l+d-1)$. As a result, $c_l(f) > 0$ for $l \in \mathbb{N}_0$, hence f is SPD on $\mathbb{S}^d$ for odd $d \geq 3$.

First we prove a lemma relating Gegenbauer polynomials to spherical Bessel functions. For $\alpha \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$, let

$$\hat{j}_\alpha(x) = \frac{2^{\alpha+1}\Gamma(\alpha+\frac{3}{2})}{\Gamma(\frac{1}{2})}\frac{j_\alpha(x)}{x^\alpha}, \tag{2.1}$$

be the normalized spherical Bessel function (10.47 in [15]) with $\hat{j}_\alpha(0) = 1$,

$$\hat{C}_l^{(\alpha)}(x) = C_l^{(\alpha)}(x)/C_l^{(\alpha)}(1),$$

be the normalized Gegenbauer polynomial, and $B_{l,\alpha}(x)$ be the B-spline function of order α with knots $l^2, (l+2)^2, \ldots, (l+2\alpha+2)^2$, normalized by $\int_0^\infty B_{l,\alpha}(x)dx = 1$. The explicit form is

$$B_{l,\alpha}(x) = (\alpha+1)\sum_{k=0}^{\alpha+1}\frac{((l+2k)^2 - x)_+^\alpha}{\omega'((l+2k)^2)}, \tag{2.2}$$

where $\omega(x) = \prod_{k=0}^{\alpha+1}(x - (l+2k)^2)$. As shown by [4], for $\alpha \geq 1$, the B-spline function is continuous, positive between the first and last knots and vanishing elsewhere, and satisfying

$$B'_{l,\alpha}(x) = (\alpha+1)\frac{B_{l,\alpha-1}(x) - B_{l+2,\alpha-1}(x)}{(l+2\alpha+2)^2 - l^2} = \frac{B_{l,\alpha-1}(x) - B_{l+2,\alpha-1}(x)}{4(l+\alpha+1)}. \tag{2.3}$$

Lemma 2.1. *For $\alpha \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$,*

$$\hat{C}_l^{(\alpha+1)}(\cos x)(\sin x)^{2\alpha+2} = \int_0^\infty \hat{j}_\alpha(\sqrt{p}x)x^{2\alpha+2}B_{l,\alpha}(p)dp. \tag{2.4}$$

Proof. We prove Eq. (2.4) by induction on α. By Eqs. (2.1) and (2.2),

$$\hat{j}_0(x) = \frac{\sin x}{x}, \quad \hat{C}_l^{(1)}(x) = \frac{\sin((l+1)x)}{(l+1)\sin(x)}, \quad B_{l,0} = \frac{1}{4(l+1)}\chi_{[l^2,(l+2)^2]}.$$

Eq. (2.4) holds for $\alpha = 0$. From the identity $-(2\alpha+1)\hat{j}'_{\alpha-1}(x) = x\hat{j}_\alpha(x)$ (10.51.3 in [15]), we have

$$-2(2\alpha+1)\frac{\partial}{\partial p}\left(\hat{j}_{\alpha-1}(\sqrt{p}x)x^{2\alpha}\right) = \hat{j}_\alpha(\sqrt{p}x)x^{2\alpha+2}. \tag{2.5}$$

The normalized Gegenbauer polynomials satisfy (18.9.8 in [15])

$$(2\alpha+1)(\hat{C}_l^{(\alpha)}(\cos x) - \hat{C}_{l+2}^{(\alpha)}(\cos x)) = 2(l+\alpha+1)\hat{C}_l^{(\alpha+1)}(\cos x)(\sin x)^2. \tag{2.6}$$

For $\alpha \geq 1$, assume that Eq. (2.4) holds for $\alpha - 1$, i.e.,

$$\hat{C}_l^{(\alpha)}(\cos x)(\sin x)^{2\alpha} = \int_0^\infty \hat{j}_{\alpha-1}(\sqrt{p}x)x^{2\alpha}B_{l,\alpha-1}(p)dp. \tag{2.7}$$

Using Eqs. (2.5), (2.3), (2.7), and (2.6) in that order,

$$\begin{aligned}\int_0^\infty \hat{j}_\alpha(\sqrt{p}x)x^{2\alpha+2}B_{l,\alpha}(p)dp &= -2(2\alpha+1)\int_0^\infty \frac{\partial}{\partial p}\left(\hat{j}_{\alpha-1}(\sqrt{p}x)x^{2\alpha}\right)B_{l,\alpha}(p)dp \\ =&2(2\alpha+1)\int_0^\infty \hat{j}_{\alpha-1}(\sqrt{p}x)x^{2\alpha}\frac{B_{l,\alpha-1}(p) - B_{l+2,\alpha-1}(p)}{4(l+\alpha+1)}dp \\ =&\frac{(2\alpha+1)}{2(l+\alpha+1)}\left(\hat{C}_l^{(\alpha)}(\cos x) - \hat{C}_{l+2}^{(\alpha)}(\cos x)\right)(\sin x)^{2\alpha} = \hat{C}_l^{(\alpha+1)}(\cos x)(\sin x)^{2\alpha+2}.\end{aligned}$$

□

From Lemma 2.1 we can obtain the following theorem.

Theorem 2.2. *For odd $d \geq 3$, a PD function f on $\mathbb{R}^d$ with support in $[0,\pi]$ has positive spectral coefficients on $\mathbb{S}^d$ and $\mathbb{RP}^d$, and thus is SPD on $\mathbb{S}^d$ and $\mathbb{RP}^d$.*

Proof. For $l \in \mathbb{N}_0$, the l'th spectral coefficient of $f(x)$ on $\mathbb{S}^d$ is

$$c_l(f) = \omega_{d-1}\int_0^\pi f(x)\hat{C}_l^\alpha(\cos x)(\sin x)^{2\alpha}dx,$$

with $\alpha = (d-1)/2 \in \mathbb{N}$ and $\omega_{d-1} = \mu(\mathbb{S}^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$. By Lemma 2.1,

$$\int_0^\pi f(x)\hat{C}_l^\alpha(\cos x)(\sin x)^{2\alpha}dx = \int_0^\pi\int_{l^2}^{(l+2\alpha)^2} f(x)\hat{j}_{\alpha-1}(\sqrt{p}x)x^{2\alpha}B_{l,\alpha-1}(p)dpdx. \tag{2.8}$$

Since $f(\cdot)$ is PD on $\mathbb{R}^d$ with support in $[0,\pi]$, the radial part of its Fourier transform

$$F(k) = \omega_{d-1}\int_0^\pi f(x)\hat{j}_{\alpha-1}(kx)x^{2\alpha}dx$$

is nonnegative and not vanishing on an open set. We can rewrite Eq. (2.8) as

$$c_l(f) = \int_{l^2}^{(l+2\alpha)^2} F(\sqrt{p})B_{l,\alpha-1}(p)dp.$$

As $B_{l,\alpha-1}(p)$ is positive on $(l^2,(l+2\alpha)^2)$, $c_l(f) > 0$. Consequently, f is SPD on $\mathbb{S}^d$. The spectral coefficients of f on $\mathbb{RP}^d$ for $l \in \mathbb{N}_0$ were given in [10] as

$$\int_0^\pi f(x)p_l^{(\frac{d-2}{2},-\frac{1}{2})}(\cos x)(\sin\frac{x}{2})^{d-1}dx = 2\int_0^{\pi/2} f(2x)\hat{C}_{2l}^{\frac{d-1}{2}}(\cos x)(\sin x)^{d-1}dx, \tag{2.9}$$

where $p_l^{(\alpha,\beta)}(x) = P_l^{(\alpha,\beta)}(x)/P_l^{(\alpha,\beta)}(1)$ is the normalized Jacobi polynomial. Since $g(x) \equiv f(2x)$ is PD on $\mathbb{R}^d$ with support in $[0, \pi/2]$, the spectral coefficients of f on $\mathbb{RP}^d$ are positive. By [1], $f(x)$ is SPD on $\mathbb{RP}^d$. □

3. Truncated power functions

[8] proved that the truncated power function $f(\cdot; t) = (t - \cdot)_+^{(d+1)/2}$ is PD on $\mathbb{R}^d$ for any $t > 0$. By Theorem 2.2, $f(\cdot; t)$ is SPD on $\mathbb{S}^d$ for odd $d \geq 3$ if $t \leq \pi$. To prove that $f(\cdot; t)$ is SPD on all spheres with $d \geq 2$, we introduce the concept of positive mixture. For a real-valued continuous function $f(\cdot)$ on $[0, a]$ with $a > 0$ and a family of functions $g(\cdot; t)$ supported on $[0, a]$ with $t \in D \subset \mathbb{R}$, $f(\cdot)$ is a positive mixture of $g(\cdot; t)$ with $t \in D$ if

$$f(x) = \int_D g(x; t) d\mu(t),$$

where $\mu(t)$ is a positive measure, i.e., $\mu(U) > 0$ for any nonempty open set $U \subset D$. If $g(\cdot; t)$ is PD (SPD) on a given space for $t \in D$, and $f(\cdot)$ is positive mixture of $g(\cdot; t)$ with $t \in D$, then $f(\cdot)$ is PD (SPD) on the same space, because for any n distinct points $\{\mathbf{x}_i\}_1^n$ in the given space and n real numbers $\{a_i\}_1^n$ that are not all zero,

$$\sum_{i=1}^n \sum_{j=1}^n f(\rho(\mathbf{x}_i, \mathbf{x}_j)) a_i a_j = \int_D \left(\sum_{i=1}^n \sum_{j=1}^n g(\rho(\mathbf{x}_i, \mathbf{x}_j); t) a_i a_j \right) d\mu(t) \geq 0$$

for a PD function and strict inequality for a SPD function. In other words, PD and SPD properties are preserved under positive mixtures.

For $t > 0$, denote by χ_t the characteristic function on $[0, t]$. Let

$$f_d(\mathbf{x}, \mathbf{y}; t) = \int_{\mathbb{R}^d} \chi_t(\|\mathbf{x} - \mathbf{z}\|) \chi_t(\|\mathbf{y} - \mathbf{z}\|) d\mathbf{z},$$

be the self convolution of χ_t on $\mathbb{R}^d$, and denote its isotropic part by $f_d(r; t)$ with $r = \|\mathbf{x} - \mathbf{y}\|$. For $t \in (0, \pi/2]$, let

$$g_d(\mathbf{x}, \mathbf{y}; t) = \int_{\mathbb{S}^d} \chi_t(\rho(\mathbf{x}, \mathbf{z})) \chi_t(\rho(\mathbf{y}, \mathbf{z})) d\mathbf{z},$$

be the self convolution of χ_t on $\mathbb{S}^d$, and denote its isotropic part by $g_d(r; t)$ with $r = \rho(\mathbf{x}, \mathbf{y})$ on S^d. For any function on $\mathbb{R}^d$, the Fourier transform of its self convolution is the square of its Fourier transform. For any function on $\mathbb{S}^d$, the l'th spectral coefficient of its self convolution is the square of its l'th spectral coefficient (see the proof of Theorem 3.5). Therefore, $f_d(\cdot; t)$ is PD on $\mathbb{R}^d$, and $g_d(\cdot; t)$ is PD on $\mathbb{S}^d$. We will prove that for $t \in (0, \pi/2]$, $f_d(\cdot; t)$ is a positive mixture of $g_d(\cdot; \beta)$ with $\beta \in [0, t]$, hence is also PD on $\mathbb{S}^d$. In addition, we will show that for $t \in (0, \pi]$, the truncated power function $f(\cdot; t) = (t - \cdot)_+^{(d+1)/2}$ is a positive mixture of $f_d(\cdot; \beta)$ with $\beta \in [0, t/2]$, hence is PD on both $\mathbb{R}^d$ and $\mathbb{S}^d$.

Our proof uses the concept of completely monotonic (CM) functions introduced in [3]. For $a > 0$ or $a = \infty$, let $f(\cdot)$ be a real valued C^∞ function on $(0, a)$. The function $f(\cdot)$ is CM on $(0, a)$ if

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x \in (0, a)$ and $n \in \mathbb{N}_0$. By the Hausdorff–Bernstein–Widder theorem [3], a function is CM on $(0, \infty)$ if and only if its inverse Laplace transform is nonnegative. As shown in [3], for a finite a, f is CM on $(0, a)$ if and only if there exist $b_n \geq 0$ for $n \in \mathbb{N}_0$ such that

$$f(x) = \sum_{n=0}^{\infty} b_n (a-x)^n, \tag{3.1}$$

which converges for $x \in (0, a)$. We start with a few lemmas about CM functions and positive mixtures.

Lemma 3.1. *Let f be a smooth function on $(0, a]$ with $a > 0$.*

(i) If f is CM on $(0, a)$, $(f(\cdot) - f(a))/(a - \cdot)$ is CM on $(0, a)$.

(ii) If $f(0+) \geq 0$, $f(a) = 0$, $f'(a) < 0$, and $-f''$ is CM on $(0, a)$, $(a - \cdot)/f(\cdot)$ is CM on $(0, a)$.

(iii) If $f(0+) \geq 0$, $f \not\equiv 0$, and f' is CM on $(0, a)$, $1/f(\cdot)$ is CM on $(0, a)$.

(iv) If $f(a) > 0$ and $-(\ln f)'$ is CM on $(0, a)$, $f^{\lambda}(\cdot)$ is CM on $(0, a)$ for $\lambda > 0$.

Statements (iii) and (iv) also hold for $a = \infty$.

Proof. (i) By Eq. (3.1), $(f(x) - f(a))/(a - x) = \sum_{n=0}^{\infty} b_{n+1}(a - x)^n$ with $b_n \geq 0$ for $n \in \mathbb{N}$, hence $(f(\cdot) - f(a))/(a - \cdot)$ is CM on $(0, a)$.

(ii) By Eq. (3.1) and the given conditions, $f(x) = -f'(a)(a-x)(1-\sum_{n=1}^{\infty} c_n(a-x)^n)$ with $c_n \geq 0$ for $n \in \mathbb{N}$, and $0 \leq \sum_{n=1}^{\infty} c_n(a - x)^n < 1$ for $x \in (0, a)$. Then

$$\frac{a-x}{f(x)} = \frac{1}{-f'(a)} \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} c_n (a-x)^n \right)^k,$$

which converges on $(0, a)$. Therefore, $(a - \cdot)/f(\cdot)$ is CM on $(0, a)$.

(iii) Since $f(a) > 0$ and $f(a) - f(\cdot)$ is CM on $(0, a)$, $1/f(\cdot) = 1/(f(a) - (f(a) - f(\cdot)))$ is CM on $(0, a)$.

(iv) Since $\ln f(\cdot) - \ln f(a)$ is CM on $(0, a)$, $f^{\lambda}(\cdot) = f^{\lambda}(a) \exp[\lambda(\ln f(\cdot) - \ln f(a))]$ is CM on $(0, a)$. □

Lemma 3.2. *(i) The function $\arctan(\sqrt{\cdot})/\sqrt{\cdot}$ is CM on $(0, \infty)$.*

(ii)

$$F(s) = \left(\frac{\pi}{2}\right)^2 - \arctan^2(\sqrt{s}) - \frac{\arctan\sqrt{s}}{\sqrt{s}}. \tag{3.2}$$

$F(\cdot)$ is CM on $(0, \infty)$.

(iii)

$$G(s) = \frac{\pi}{2\sqrt{s}} - \frac{\arctan\sqrt{s}}{\sqrt{s}} - \frac{1}{1+s}. \tag{3.3}$$

$G(\cdot)$ is CM on $(0, \infty)$.

Proof. (i) By the identity

$$\int_0^{\infty} \int_{\sqrt{t}}^{\infty} \frac{e^{-x^2-st}}{\sqrt{t}} dx dt = \frac{2}{\sqrt{s}} \int_0^{\infty} \int_{y/\sqrt{s}}^{\infty} e^{-x^2-y^2} dx dy = \frac{\arctan(\sqrt{s})}{\sqrt{s}},$$

$$\mathcal{L}^{-1}\left(\frac{\arctan\sqrt{s}}{\sqrt{s}}\right) = \frac{\sqrt{\pi}}{2} \frac{\operatorname{erfc}(\sqrt{t})}{\sqrt{t}} > 0,$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform and $\operatorname{erfc}(\cdot)$ is the complementary error function. Therefore, $\arctan(\sqrt{\cdot})/\sqrt{\cdot}$ is CM on $(0, \infty)$.

(ii) By the identity

$$\left(\frac{\pi}{2}\right)^2 - \arctan^2(\sqrt{s}) = \int_s^\infty \frac{\arctan\sqrt{x}}{\sqrt{x}} \frac{1}{1+x} dx,$$

we have

$$\mathcal{L}^{-1}(F) = \frac{\sqrt{\pi}}{2}\frac{e^{-t}}{t}\int_0^t \frac{e^x}{\sqrt{x}}\operatorname{erfc}(\sqrt{x})dx - \frac{\sqrt{\pi}}{2}\frac{\operatorname{erfc}(\sqrt{t})}{\sqrt{t}}.$$

Since

$$2x^{3/2}\left(\frac{e^x}{\sqrt{x}}\operatorname{erfc}(\sqrt{x})\right)' = e^x\operatorname{erfc}(\sqrt{x})(2x-1) - \frac{2\sqrt{x}}{\sqrt{\pi}} < e^x\operatorname{erfc}(\sqrt{x})2x - \frac{2\sqrt{x}}{\sqrt{\pi}} < 0,$$

$\mathcal{L}^{-1}(F) > 0$, so $F(\cdot)$ is CM on $(0, \infty)$.

(iii)

$$\mathcal{L}^{-1}(G) = \frac{\sqrt{\pi}}{2}\frac{\operatorname{erf}(\sqrt{t})}{\sqrt{t}} - e^{-t} = \frac{1}{\sqrt{t}}\int_0^{\sqrt{t}} e^{-x^2}dx - e^{-t} > 0.$$

Therefore, $G(\cdot)$ is CM on $(0, \infty)$. □

Lemma 3.3. *Let $\beta > 0$, $\lambda > 0$, and*

$$h(x;\beta) = \left(\beta^2 - \arctan^2(\sqrt{x})\right)_+.$$

(i) For $\nu > 0$, $(a-\cdot)_+^{\lambda+\nu}$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t \in [0, a]$.

(ii) If a non-constant $f(\cdot)$ is CM on $(0, a)$, $(a-\cdot)_+^\lambda f(\cdot)$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t \in [0, a]$.

(iii) For $0 < \beta < \pi/2$, $h^\lambda(\cdot;\beta)$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t \in [0, \tan^2\beta]$.

(iv) For $\beta \geq \pi/2$, $h^\lambda(\cdot;\beta)$ and $(\beta - \arctan(\sqrt{\cdot}))^\lambda$ are CM on $(0, \infty)$.

Proof. (i) For $\lambda > -1$ and $\nu > 0$, by setting $y = (t-x)/(1-x)$,

$$\int_0^1 (t-x)_+^\lambda (1-t)^{\nu-1}dt = (1-x)_+^{\lambda+\nu}\int_0^1 y^\lambda(1-y)^{\nu-1}dy = (1-x)_+^{\lambda+\nu}B(\lambda+1,\nu),$$

where $B(x, y)$ is the Beta function. For $\lambda > -1$ and $\nu > 0$, $B(\lambda+1, \nu) > 0$, so $(1-\cdot)_+^{\lambda+\nu}$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t \in [0, 1]$. By scaling, $(a-\cdot)_+^{\lambda+\nu}$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t \in [0, a]$.

(ii) By Eq. (3.1),

$$(a-x)_+^\lambda f(x) = b_0(a-x)_+^\lambda + \sum_{n=1}^\infty b_n(a-x)_+^{\lambda+n},$$

with $b_n \geq 0$ for $n \in \mathbb{N}_0$, and $\{b_n\}_1^\infty$ is not all zero. By part (i), $(a-\cdot)_+^{\lambda+n}$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t \in [0, a]$. With a point mass of b_0 at $t = a$, $(a-\cdot)_+^\lambda f(\cdot)$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t \in [0, a]$.

(iii) For $0 < \beta < \pi/2$ and $x \in [0, \tan^2\beta]$,

$$h^\lambda(x;\beta) = \left(\frac{\beta(\tan^2\beta - x)}{\tan\beta(1+\tan^2\beta)}\right)^\lambda g^\lambda(x;\beta),$$

where

$$g(x;\beta)=\frac{\tan\beta(1+\tan^2\beta)(\beta^2-\arctan^2\sqrt{x})}{\beta(\tan^2\beta-x)}.$$

We have

$$-(\ln g(x;\beta))'=\frac{1}{\tan^2\beta-x}\left[\frac{\arctan\sqrt{x}}{\sqrt{x}}\frac{\tan^2\beta-x}{f(x;\beta)}-1\right],$$

where

$$f(x;\beta)=\left(\beta^2-\arctan^2\sqrt{x}\right)(1+x). \tag{3.4}$$

By Lemma 3.2(i), $(\arctan\sqrt{\cdot})/\sqrt{\cdot}$ is CM on $(0,\tan^2\beta)$. We have

$$f'(x;\beta)=\beta^2-\arctan^2\sqrt{x}-\frac{\arctan\sqrt{x}}{\sqrt{x}}=F(x)+\beta^2-\left(\frac{\pi}{2}\right)^2,$$

with $F(\cdot)$ defined in Eq. (3.2). By Lemma 3.2(ii), $-f''(\cdot;\beta)=-F'(\cdot)$ is CM on $(0,\tan^2\beta)$. We also have

$$f(0;\beta)=\beta^2,\quad f(\tan^2\beta;\beta)=0,\quad f'(\tan^2\beta;\beta)=-\beta/\tan\beta.$$

By Lemma 3.1(ii), $(\tan^2\beta-\cdot)/f(\cdot;\beta)$ is CM on $(0,\tan^2\beta)$, and so is the function $f(\cdot)$ defined by

$$f(x)\equiv\frac{\arctan\sqrt{x}}{\sqrt{x}}\frac{\tan^2\beta-x}{f(x;\beta)}.$$

Applying l'Hôspital's rule, $f(\tan^2\beta)=1$. By Lemma 3.1(i) with $a=\tan^2\beta$, $-(\ln g(\cdot;\beta))'$ is CM on $(0,\tan^2\beta)$. Since $g(\tan^2\beta;\beta)=1$, by Lemma 3.1(iv), $g^\lambda(\cdot;\beta)$ is CM on $(0,\tan^2\beta)$. Since $g(\cdot;\beta)$ is not a constant, by part (i), $h^\lambda(\cdot;\beta)$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t\in[0,\tan^2\beta]$.

(iv) For $\beta\geq\pi/2$ and $x\in[0,\infty)$,

$$-(\ln h(x;\beta))'=\frac{\arctan\sqrt{x}}{\sqrt{x}f(x;\beta)},$$

with f given by Eq. (3.4). By Lemma 3.2(ii), $f'(\cdot;\beta)$ is CM on $(0,\infty)$. By Lemma 3.1(iii), $1/f(\cdot;\beta)$ is CM on $(0,\infty)$. Combined with Lemma 3.2(i), $-(\ln h(\cdot;\beta))'$ is CM on $(0,\infty)$. By Lemma 3.1(iv), $h^\lambda(\cdot;\beta)$ is CM on $(0,\infty)$. For $\beta\geq\pi/2$ and $x\in[0,\infty)$,

$$-(\ln(\beta-\arctan\sqrt{x}))'=\frac{1}{\phi(x;\beta)(1+x)},$$

with $\phi(x;\beta)=2\sqrt{x}(\beta-\arctan\sqrt{x})$. Since $\phi'(x;\beta)=(\beta-\pi/2)/\sqrt{x}+G(x)$, with $G(\cdot)$ defined in Eq. (3.3), by Lemma 3.2(iii) and that $1/\sqrt{\cdot}$ is CM on $(0,\infty)$, $\phi'(\cdot;\beta)$ is CM on $(0,\infty)$. Since $\phi(0;\beta)=0$, by Lemma 3.1(iii), $1/\phi(\cdot;\beta)$ is CM on $(0,\infty)$. Since $1/(1+\cdot)$ is also CM on $(0,\infty)$, by Lemma 3.1(iv), $(\beta-\arctan\sqrt{\cdot})^\lambda$ is CM on $(0,\infty)$. □

Lemma 3.4. *For $\lambda>0$ and $0<\nu<1$, $(1-\cdot^\nu)_+^\lambda$ is a positive mixture of $(t-\cdot)_+^\lambda$ with $t\in[0,1]$.*

Proof. For $x\in[0,1]$,

$$(1-x^\nu)^\lambda=(\nu(1-x))^\lambda g^\lambda(x),$$

where $g(\cdot)$ is a continuous function on $[0, 1]$,

$$g(x) = \begin{cases} \frac{1-x^\nu}{\nu(1-x)}, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} .$$

For $0 < \nu < 1$, by the binomial expansion about $x = 1$, $x^{1-\nu} - x = \nu(1-x)(1-\sum_{k=1}^\infty a_n(1-x)^n)$ with $a_n > 0$ for $n \ge 1$. Subsequently, using the identity $(1-y)^{-1} = \sum_{k=0}^\infty y^k$ for $y = \sum_{n=1}^\infty a_n(1-x)^n$,

$$-(\ln g(x))' = \frac{\nu}{x^{1-\nu} - x} - \frac{1}{1-x} = \sum_{n=0}^\infty b_n(1-x)^n,$$

with $b_n > 0$ for $n \in \mathbb{N}_0$. Therefore, $-(\ln g(\cdot))'$ is CM on $(0, 1)$. By Lemma 3.1(iv), $g^\lambda(\cdot)$ is CM on $(0, 1)$ for $\lambda > 0$. By Lemma 3.3(ii), $(1 - \cdot^\nu)_+^\lambda$ is a positive mixture of $(t - \cdot)_+^\lambda$ with $t \in [0, 1]$. □

The lemmas lead to the following theorem, which incorporates the conjecture in [2].

Theorem 3.5. *For $\beta > 0$, $r \ge 0$, and $\lambda \ge 0$, let*

$$F_\lambda(r; \beta) = \int_{r/2}^\infty (\beta^2 - x^2)_+^\lambda dx,$$

$$P_\lambda(r; \beta) = \int_{r/2}^\infty (\beta - x)_+^\lambda dx = \frac{1}{\lambda+1}(\beta - \frac{r}{2})_+^{\lambda+1}.$$

(i) $F_\lambda(\cdot; \beta)$ and $P_\lambda(\cdot; \beta)$ are SPD on $\mathbb{R}^d$ for $d \ge 1$ and $\lambda \ge (d-1)/2$.

(ii) $F_\lambda(\cdot; \beta)$ and $P_\lambda(\cdot; \beta)$ are SPD on $\mathbb{S}^d$ and $\mathbb{RP}^d$ for $d \ge 2$, $0 < \beta < \pi/2$, and $\lambda \ge (d-1)/2$.

(iii) $\hat{F}_\lambda(\cdot; \beta)$ and $\hat{P}_\lambda(\cdot; \beta)$ are SPD on $\mathbb{S}^d$ for $d \ge 1$, $\beta \ge \pi/2$, and $\lambda > 0$, where

$$\hat{F}_\lambda(r; \beta) = \int_{r/2}^{\pi/2} (\beta^2 - x^2)^\lambda - (\beta^2 - (\pi/2)^2)^\lambda dx = F_\lambda(r; \beta) - F_\lambda(\pi; \beta) - F_\lambda'(\pi; \beta)(r - \pi),$$

$$\hat{P}_\lambda(r; \beta) = \int_{r/2}^{\pi/2} (\beta - x)^\lambda - (\beta - \pi/2)^\lambda dx = P_\lambda(r; \beta) - P_\lambda(\pi; \beta) - P_\lambda'(\pi; \beta)(r - \pi).$$

Proof. (i) For $\beta > 0$, denote the isotropic part of the self convolution of χ_β on $\mathbb{R}^d$ by $f_d(\cdot; \beta)$. Since

$$\int_{\mathbb{R}^d} f_d(\|\mathbf{z}\|; \beta) e^{i\mathbf{k}\cdot\mathbf{z}} d\mathbf{z} = \left(\int_{\mathbb{R}^d} \chi_\beta(\|\mathbf{x}\|) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}\right)^2 \ge 0,$$

$f_d(\cdot; \beta)$ is PD on $\mathbb{R}^d$. Since χ_β is compactly supported, $f_d(\cdot; \beta)$ is SPD on $\mathbb{R}^d$. With $\|\mathbf{x} - \mathbf{y}\| = r$,

$$\begin{aligned} f_d(r; \beta) &= \mu(\{\mathbf{z} \in \mathbb{R}^d, \|\mathbf{x} - \mathbf{z}\| \le \beta, \|\mathbf{y} - \mathbf{z}\| \le \beta\}) = 2\int_{r/2}^\beta \int_{x_1^2 + \cdots + x_{d-1}^2 \le \beta^2 - x_d^2} dx_1 \cdots dx_d \\ &= 2B_{d-1} \int_{r/2}^\infty (\beta^2 - x^2)_+^{\frac{d-1}{2}} dx, \end{aligned}$$

where

$$B_d = \mu(\mathbf{B}^d) = \pi^{\frac{d}{2}} / \Gamma(\frac{d}{2} + 1).$$

Therefore, $F_{(d-1)/2}(\cdot;\beta)$ is SPD on $\mathbb{R}^d$. For $\lambda > (d-1)/2$, by Lemma 3.3(i), $(\beta^2 - \cdot^2)_+^\lambda$ is a positive mixture of $(t^2 - \cdot^2)_+^{(d-1)/2}$ with $t \in [0,\beta]$, so $F_\lambda(\cdot;\beta)$ is a positive mixture of $F_{(d-1)/2}(\cdot;t)$ with $t \in [0,\beta]$. Hence $F_\lambda(\cdot;\beta)$ is SPD on $\mathbb{R}^d$. For $\lambda = 0$, $P_\lambda(\cdot;\beta) = F_\lambda(\cdot;\beta)$. For $\lambda > 0$, by Lemma 3.4 with $\nu = 1/2$, $(1-\cdot)_+^\lambda$ is a positive mixture of $(t^2-\cdot^2)_+^\lambda$ with $t \in [0,1]$. By scaling, there is a positive measure $\mu_1(\cdot)$ on $[0,\beta]$ such that

$$(\beta - x)_+^\lambda = \int_0^\beta (t^2 - x^2)_+^\lambda d\mu_1(t).$$

Then

$$P_\lambda(r;\beta) = \int_{r/2}^\infty \int_0^\beta (t^2 - x^2)_+^\lambda d\mu_1(t)dx = \int_0^\beta F_\lambda(r;t)d\mu_1(t). \tag{3.5}$$

So $P_\lambda(\cdot;\beta)$ is a positive mixture of $F_\lambda(\cdot;t)$ with $t \in [0,\beta]$. Therefore, $P_\lambda(\cdot;\beta)$ is SPD on $\mathbb{R}^d$ if $\lambda \geq (d-1)/2$.

(ii) For $0 < \beta < \pi/2$, denote the isotropic part of the self convolution of χ_β on $\mathbb{S}^d$ by $g_d(\cdot;\beta)$. Recall the Funk–Hecke formula on $\mathbb{S}^d$ embedded in $\mathbb{R}^{d+1}$ (Lemma 2 in [11]),

$$\int_{\mathbb{S}^d} p_{\alpha,l}(\mathbf{x}\cdot\mathbf{z})p_{\alpha,l}(\mathbf{y}\cdot\mathbf{z})d\mathbf{z} = K_{d,l}\,p_{\alpha,l}(\mathbf{x}\cdot\mathbf{y}), \quad K_{d,l} \equiv \frac{\omega_d}{\dim H_l},$$

where

$$\omega_d = \mu(\mathbb{S}^d) = 2\pi^{\frac{d+1}{2}}/\Gamma(\frac{d+1}{2}),$$

and H_l is the eigenspace of the Laplacian on $\mathbb{S}^d$ corresponding to $\lambda_l = l(l+d-1)$,

$$\dim H_l = \frac{(2l+d-1)\Gamma(l+d-1)}{\Gamma(d)\Gamma(l+1)}.$$

Applying the Funk–Hecke formula to the spectral expansion of χ_β,

$$\chi_\beta(\rho(\mathbf{x},\mathbf{y})) = \sum_{l=0}^\infty \frac{\int_{\mathbb{S}^d}\chi_\beta(\rho(\mathbf{x},\mathbf{y}))p_{\alpha,l}(\mathbf{x}\cdot\mathbf{y})d\mathbf{x}}{K_{d,l}} p_{\alpha,l}(\mathbf{x}\cdot\mathbf{y}),$$

we find the l'th spectral coefficient of $g_d(\cdot;\beta)$,

$$c_l(g_d(\cdot;\beta)) = \int_{\mathbb{S}^d} g_d(\rho(\mathbf{x},\mathbf{y});\beta)p_{\alpha,l}(\mathbf{x}\cdot\mathbf{y})d\mathbf{x} = \left(\int_{\mathbb{S}^d}\chi_\beta(\rho(\mathbf{x},\mathbf{y}))p_{\alpha,l}(\mathbf{x}\cdot\mathbf{y})d\mathbf{x}\right)^2 \geq 0.$$

Therefore, $g_d(\cdot;\beta)$ is PD on $\mathbb{S}^d$. Notice that $c_l(g_d(\cdot;\beta))$ vanishes for finitely many β in $(0,\pi/2)$ because $p_{\alpha,l}(\cdot)$ is a polynomial which has finitely many zeros. With $\rho(\mathbf{x},\mathbf{y}) = r$,

$$\begin{aligned} g_d(r;\beta) &= \mu(\{\mathbf{z}\in\mathbb{S}^d, \rho(\mathbf{x},\mathbf{z}) \leq \beta, \rho(\mathbf{y},\mathbf{z}) \leq \beta\}) \\ &= 2B_{d-1}(\cos\beta)^{d-1}\int_{r/2}^{\pi/2}\left(\tan^2\beta - \tan^2 x\right)_+^{\frac{d-1}{2}} dx. \end{aligned}$$

To derive the formula above, we embed $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ as $\{\mathbf{z} = (z_0,\ldots,z_d), \|\mathbf{z}\| = 1\}$. Set $\mathbf{x} = (1,0,0,\ldots,0)$, $\mathbf{y} = (\cos(r),\sin(r),0,\ldots,0)$. Write $z_0 = \sqrt{1-t^2}\cos\phi$ and $z_1 =$

$\sqrt{1-t^2}\sin\phi$ with $t=\sqrt{z_2^2+\cdots+z_d^2}$. We have

$$g_d(r;\beta)=2\int_{\rho(x,z)\le\beta,\ r/2\le\phi\le\beta}\frac{dz_1\cdots dz_d}{z_0}=2\int_{z_0\ge\cos\beta,\ r/2\le\phi\le\beta}\frac{dz_1 d(B_{d-1}t^{d-1})}{z_0}$$

$$=2\int_{r/2}^{\beta}\int_{0\le t\le\sqrt{1-(\cos\beta/\cos\phi)^2}}d(B_{d-1}t^{d-1})d\phi$$

$$=2B_{d-1}(\cos\beta)^{d-1}\int_{r/2}^{\beta}(\sec^2\beta-\sec^2\phi)^{\frac{d-1}{2}}d\phi.$$

By Lemma 3.3(iii), $(\beta^2-\arctan^2\sqrt{\cdot})_+^{\lambda}$ is a positive mixture of $(t-\cdot)_+^{\lambda}$ with $t\in[0,\tan^2\beta]$. For $d\ge 2$, with $\lambda=(d-1)/2>0$ and a change of variable, $(\beta^2-\cdot^2)_+^{(d-1)/2}$ is a positive mixture of $(\tan^2 t-\tan^2\cdot)_+^{(d-1)/2}$ with a positive measure $\mu_2(t)$ on $[0,\beta]$. As a result,

$$f_d(r;\beta)=\int_0^{\beta}g_d(r;t)(\cos t)^{1-d}d\mu_2(t),$$

and

$$c_l(F_{(d-1)/2}(\cdot;\beta))=\frac{1}{2B_{d-1}}\int_0^{\beta}c_l(g_d(\cdot;t))(\cos t)^{1-d}d\mu_2(t)>0,$$

because $c_l(g_d(\cdot;t))>0$ except for finitely many $t\in(0,\beta)$. It implies that $F_{(d-1)/2}(\cdot;\beta)$ is SPD on $\mathbb{S}^d$. By Lemma 3.3(i), $F_\lambda(\cdot;\beta)$ is SPD on $\mathbb{S}^d$ for $\lambda\ge(d-1)/2$. By Eq. (3.5), $P_\lambda(\cdot;\beta)$ is a positive mixture of $F_\lambda(\cdot;t)$ with $t\in[0,\beta]$, so $P_\lambda(\cdot;\beta)$ is SPD on $\mathbb{S}^d$ for $\lambda\ge(d-1)/2$. By Eq. (2.9), $F_\lambda(\cdot;\beta)$ and $P_\lambda(\cdot;\beta)$ are SPD on $\mathbb{RP}^d$ for $\lambda\ge(d-1)/2$.

(iii) For $\beta\ge\pi/2$ and $\lambda>0$, by Lemma 3.3(iv),

$$(\beta^2-\arctan^2\sqrt{\cdot})^{\lambda}-(\beta^2-(\pi/2)^2)^{\lambda}\quad\text{and}\quad(\beta-\arctan\sqrt{\cdot})^{\lambda}-(\beta-\pi/2)^{\lambda}\tag{3.6}$$

are CM on $(0,\infty)$, whose inverse Laplace transforms are nonnegative. By the identity

$$e^{-y}=\frac{1}{\Gamma(\nu+1)}\int_0^{\infty}(t-y)_+^{\nu}e^{-t}dt,$$

$\exp(-\cdot)$ is a positive mixture of $(t-\cdot)_+^{\nu}$ with $t\in(0,\infty)$ for any $\nu>0$. A non-constant function $f(\cdot)$ that is CM on $(0,\infty)$ can be expressed as

$$f(x)=\int_0^{\infty}\exp(-sx)d\mu(s),$$

where $\mu(\cdot)$ is nonnegative and $\mu((0,\infty))>0$. For $s>0$ and $\nu>0$, $\exp(-sx)$ is a positive mixture of $(t-x)_+^{\nu}$ with $t\in(0,\infty)$, so $f(\cdot)$ is a positive mixture of $(t-\cdot)_+^{\nu}$ with $t\in(0,\infty)$. As a result, the two functions in Eq. (3.6) are positive mixtures of $(t-\cdot)_+^{\nu}$ with $t\in(0,\infty)$ for any $\nu>0$. With $\nu=(d-1)/2$ and changes of variables,

$$(\beta^2-x^2)^{\lambda}-(\beta^2-(\pi/2)^2)^{\lambda}=\int_0^{\pi/2}(\tan^2\theta-\tan^2 x)_+^{(d-1)/2}d\mu_3(\theta),$$

$$(\beta-x)^{\lambda}-(\beta-\pi/2)^{\lambda}=\int_0^{\pi/2}(\tan^2\theta-\tan^2 x)_+^{(d-1)/2}d\mu_4(\theta),$$

for some positive measures $\mu_3(\cdot)$ and $\mu_4(\cdot)$. Consequently,

$$\hat{F}_\lambda(\cdot;\beta)=\int_0^{\pi/2}\frac{g_d(r;\theta)}{2B_{d-1}(\cos\theta)^{d-1}}d\mu_3(\theta),\quad \hat{P}_\lambda(\cdot;\beta)=\int_0^{\pi/2}\frac{g_d(r;\theta)}{2B_{d-1}(\cos\theta)^{d-1}}d\mu_4(\theta).$$

$\hat{F}_\lambda(\cdot;\beta)$ and $\hat{P}_\lambda(\cdot;\beta)$ are positive mixtures of $g_d(\cdot;t)$ with $t \in (0,\pi/2)$, thus are SPD on $\mathbb{S}^d$. □

For $\beta \geq \pi/2$, $d \geq 1$, and $\lambda > 0$, since $F_\lambda(\pi;\beta) \geq 0$, $F'_\lambda(\pi;\beta) \leq 0$, and $(\pi - \cdot)$ is PD on $\mathbb{S}^d$, $F_\lambda(\pi;\beta)+F'_\lambda(\pi;\beta)(\cdot-\pi)$ is PD on $\mathbb{S}^d$. Since $F_\lambda(r;\beta) = \hat{F}_\lambda(r;\beta)+F_\lambda(\pi;\beta)+F'_\lambda(\pi;\beta)(r-\pi)$ and $\hat{F}_\lambda(\cdot;\beta)$ is SPD on $\mathbb{S}^d$, $F_\lambda(\cdot;\beta)$ is SPD on $\mathbb{S}^d$. Similarly, $P_\lambda(\pi;\beta) \geq 0$, $P'_\lambda(\pi;\beta) \leq 0$, and $\hat{P}_\lambda(\cdot;\beta)$ is SPD on $\mathbb{S}^d$, so $P_\lambda(r;\beta) = \hat{P}_\lambda(r;\beta) + P_\lambda(\pi;\beta) + P'_\lambda(\pi;\beta)(r-\pi)$ is SPD on $\mathbb{S}^d$. It leads to the following corollary analogous to Theorem 6 in [9].

Corollary 3.5.1. *For $d \geq 2$ and $x \in [0,\infty)$, suppose $\phi(x) = \int_0^\infty (t-x)_+^{(d+1)/2} d\mu(t)$ with nonnegative (and nonzero) measure μ. Then ϕ is SPD on $\mathbb{R}^d$ and the restriction of ϕ on $[0,\pi]$ is SPD on $\mathbb{S}^d$.*

4. Remarks and conclusion

Remark 1. Theorem 2.2 does not hold for $d = 1$. A PD function on $\mathbb{R}$ with support in $[0,\pi]$ is PD on $\mathbb{S}^1$. However, it is not necessarily SPD on $\mathbb{S}^1$. For example, by [13], $(r\pi - \cdot)_+$ is SPD on $\mathbb{S}^1$ if and only if r is an irrational number between 0 and 1.

Remark 2. Theorem 3.5(iii) does not hold for $\lambda = 0$. For $\beta = \pi/2$, the function $(\pi - \cdot)_+$ is PD on $\mathbb{S}^d$ for all $d \geq 1$. However, since its spectral coefficients on $\mathbb{S}^d$ for even l's vanish except for $l = 0$, $(\pi - \cdot)_+$ is not SPD on $\mathbb{S}^d$.

In this paper, we revealed a connection between Gegenbauer polynomials and spherical Bessel functions by B-splines, and used it to prove that a PD function on $\mathbb{R}^d$ with support in $[0,\pi]$ is SPD on $\mathbb{S}^d$ and $\mathbb{RP}^d$ for odd $d \geq 3$. Using completely monotonic functions and positive mixtures, we proved that two families of compactly supported functions, one being $F_\lambda(\cdot;t/2)$ in Theorem 3.5, the other being the truncated power function $(t-\cdot)_+^{\lambda+1}$, have positive spectral coefficients on $\mathbb{S}^d$ and $\mathbb{RP}^d$ for $d \geq 2$, $\lambda \geq (d-1)/2$, and $0 < t \leq \pi$. We also showed that $F_\lambda(\cdot;t/2)$ and $(t-\cdot)_+^{\lambda+1}$ restricted to $[0,\pi]$ are SPD on all spheres for $\lambda > 0$ and $t \geq \pi$.

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Data availability

No data was used for the research described in the article.

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