Statistics and Probability Letters 226 (2025) 110520

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Positive definite radial functions on a domain

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ARTICLE INFO

MSC:
43A35
60G60

Keywords:
Positive definite radial function on balls
Compact two-point homogeneous space

ABSTRACT

In this paper, we studied continuous radial functions that are positive definite on a domain D in the Euclidean space $\mathbb{R}^d$ or a compact two-point homogeneous space $\mathbb{M}^d$. We showed that for $D \subset \mathbb{R}^d$ that contains d-balls of arbitrary radius, a radial function that is PD on D is PD on $\mathbb{R}^d$. On the other hand, for any closed proper subset $D \subset \mathbb{M}^d$, there exists a radial function that is PD on D but not PD on $\mathbb{M}^d$. We derived some sufficient conditions in terms of spectral coefficients for a continuous radial function that is PD on D to be PD on $\mathbb{M}^d$. As an example, we explicitly constructed radial functions that are PD on the unit ball embedded in the unit sphere $\mathbb{S}^d$ by a distance preserving map, but not PD on $\mathbb{S}^d$.

1. Introduction

Positive definite (PD) radial functions play an important role in the theory of isotropic Gaussian random fields on Euclidean spaces and homogeneous spaces, since they are the covariance functions of the random fields. PD radial functions on $\mathbb{R}^d$ and $\mathbb{S}^d$ were characterized by Bochner's theorem and Schoenberg (1942), respectively. A d-dimensional compact two-point homogeneous space $\mathbb{M}^d$ is a compact Riemannian manifold $\mathbb{M}^d$ with geodesic distance $\rho(\cdot,\cdot)$ such that for any two point pairs $(\mathbf{x}_1,\mathbf{x}_2)$ and $(\mathbf{y}_1,\mathbf{y}_2)$ in $\mathbb{M}^d$ with $\rho(\mathbf{x}_1,\mathbf{x}_2) = \rho(\mathbf{y}_1,\mathbf{y}_2)$ there exists an isometry $\phi(\cdot)$ on $\mathbb{M}^d$ such that $\phi(\mathbf{x}_1) = \mathbf{y}_1$, $\phi(\mathbf{x}_2) = \mathbf{y}_2$. According to Wang (1952), $\mathbb{M}^d$ falls into one of the five categories: the unit spheres $\mathbb{S}^d (d = 1, 2, \ldots)$, the real projective spaces $\mathbb{P}^d(\mathbb{R}) (d = 2, 3, \ldots)$, the complex projective spaces $\mathbb{P}^d(\mathbb{C}) (d = 4, 6, \ldots)$, the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H}) (d = 8, 12, \ldots)$, and the octonionic plane $\mathbb{P}^{16}(\mathbb{O})$. We normalize the length of a geodesic line in $\mathbb{M}^d$ to be 2π. The connected component of the group of isometries of $\mathbb{M}^d$ was given in Table 1 in Ma and Malyarenko (2020). Gangolli (1967) characterized PD radial functions on $\mathbb{M}^d$. In this paper, we study radial functions that are PD on a subset D of $\mathbb{R}^d$ or $\mathbb{M}^d$. A kernel $K(\cdot,\cdot)$ on D is PD if for any set of N points $\{\mathbf{x}_n\}_1^N$ in D and N real numbers $\{a_n\}_1^N$, $\sum_{i=1}^N \sum_{j=1}^N a_i a_j K(\mathbf{x}_i,\mathbf{x}_j) \geq 0$. A radial function f that maps $\mathbb{R}_+$ (for $\mathbb{R}^d$) or $[0,\pi]$ (for $\mathbb{M}^d$) to $\mathbb{R}$ is PD on D if for any set of N points $\{\mathbf{x}_n\}_1^N$ in D and N real numbers $\{a_n\}_1^N$,

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j f(\rho(\mathbf{x}_i,\mathbf{x}_j)) \geq 0. \tag{1.1}$$

For an isotropic function $F(\cdot)$ on $\mathbb{R}^d$, i.e., $F(\mathbf{x})$ only depends on $\|\mathbf{x}\|$, there is a unique function $f(\cdot)$ on $[0,\infty)$ such that $F(\mathbf{x}) = f(\|\mathbf{x}\|)$ for all $\mathbf{x} \in \mathbb{R}^d$. The function $f(\cdot)$ is called the radial part of $F(\cdot)$. For an isotropic kernel $K(\cdot,\cdot)$ on a metric space, i.e., $K(\mathbf{x},\mathbf{y})$ only depends on $\rho(\mathbf{x},\mathbf{y})$, the radial part of $K(\cdot,\cdot)$ is the unique function $f(\cdot)$ such that $K(\mathbf{x},\mathbf{y}) = f(\rho(\mathbf{x},\mathbf{y}))$ for all $\mathbf{x}$ and $\mathbf{y}$ in the metric space. By Bochner's theorem, a continuous radial function is PD on $\mathbb{R}^d$ if and only if it is the radial part of the Fourier transform of a positive finite isotropic Borel measure on $\mathbb{R}^d$, namely, $\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{x}} d\mu(\mathbf{k})$ where $\mu(\mathbf{k})$ only depends on $\|\mathbf{k}\|$. Rudin (1970) showed that a continuous radial function on $[0,2r]$ that is PD on the ball $\mathbb{B}_r^d(\mathbf{c}) = \{\mathbf{x} \in \mathbb{R}^d, \|\mathbf{x}-\mathbf{c}\| \leq r\}$ can be extended to

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https://doi.org/10.1016/j.spl.2025.110520
Received 20 March 2025; Received in revised form 7 June 2025; Accepted 29 July 2025
Available online 6 August 2025

a continuous radial function on $[0, \infty)$ that is PD on $\mathbb{R}^d$. We will show that a continuous radial function on $[0, \infty)$ that is PD on a cone or a quadrant is always PD on $\mathbb{R}^d$.

A continuous function $f(\cdot)$ on $[0, \pi]$ has the spectral expansion on $\mathbb{M}^d$,

$$f(\theta) = \sum_{l=0}^{\infty} c_l(f(\cdot)) p_l^{(\alpha,\beta)}(\cos(\theta)),$$

where $p_l^{(\alpha,\beta)}(x)$ is the normalized Jacobi polynomial $P_l^{(\alpha,\beta)}(x)/P_l^{(\alpha,\beta)}(1)$ (see Olver et al.) with $\alpha = (d-2)/2$, $\beta = \alpha$ for spheres and $\beta = -1/2$, 0, 1 or 3 for projective spaces; see Gangolli (1967) for details. The spectral coefficient $c_l(f(\cdot))$ is given by (see Ma and Malyarenko (2020))

$$c_l(f(\cdot)) = \frac{\dim H_l}{\omega_d} \int_{\mathbb{M}^d} f(\rho(\mathbf{x}, \mathbf{y})) p_l^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{y})) d\mathbf{x},$$

where

$$\omega_d = \mathrm{vol}(\mathbb{M}^d) = \frac{(4\pi)^{\alpha+1}\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$

H_l is the eigenspace of the Laplace–Beltrami operator Δ on $\mathbb{M}^d$ associated with $\lambda_l = l(l+\alpha+\beta+1)$, and

$$\dim H_l = \frac{(2l+\alpha+\beta+1)\Gamma(\beta+1)\Gamma(l+\alpha+\beta+1)\Gamma(l+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)\Gamma(l+1)\Gamma(l+\beta+1)}.$$

The integral is independent of $\mathbf{y} \in \mathbb{M}^d$ because the group of isometries of $\mathbb{M}^d$ acts transitively on $\mathbb{M}^d$. Koornwinder (1973) proved the addition theorem on $\mathbb{M}^d$,

$$p_l^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{y})) = \sum_{m=1}^{\dim H_l} \frac{\omega_d}{\dim H_l} S_{lm}(\mathbf{x}) S_{lm}(\mathbf{y}), \tag{1.2}$$

where $S_{lm}(\cdot)$ are the orthonormal eigenfunctions in H_l. Gangolli (1967) showed that a continuous radial function $f(\cdot)$ is PD on $\mathbb{M}^d$ if and only if its spectral coefficients $c_l(f(\cdot))$ are nonnegative and summable. We will show that a radial function that is PD on a subset of $\mathbb{M}^d$ is not necessarily PD on $\mathbb{M}^d$.

As an example in $\mathbb{M}^d$, we consider the unit ball embedded in the unit sphere by a distance preserving map. Following (Lu et al., 2020), we equip the unit ball in $\mathbb{R}^d$ with the metric,

$$\rho(\mathbf{x}, \mathbf{y}) = \arccos(\mathbf{x} \cdot \mathbf{y} + \sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2}),$$

which is isomorphic to the upper hemisphere $\mathbb{S}_+^d = \{\mathbf{x} \in \mathbb{R}^{d+1}, \|\mathbf{x}\| = 1, x_1 \geq 0\}$ by mapping $\mathbf{x}$ in the unit ball in $\mathbb{R}^d$ to $(\mathbf{x}, \sqrt{1 - \|\mathbf{x}\|^2}) \in \mathbb{S}_+^d$. Buhmann and Xu (2024) showed that if a continuous radial function is an even or odd function of $\cos(\cdot)$ on $[0, \pi]$, it is PD on the embedded ball if and only if it is PD on $\mathbb{S}^d$. We will construct a radial function that is PD on $\mathbb{S}_+^d$ but not PD on $\mathbb{S}^d$.

The rest of the paper is organized as follows. In Section 2, we investigate radial functions that are PD on a domain in $\mathbb{R}^d$ or $\mathbb{M}^d$. In Section 3, we study radial functions that are PD on the unit ball embedded in $\mathbb{S}^d$. We present the proofs in Section 4 and draw conclusions in Section 5.

2. Radial functions that are PD on a domain in $\mathbb{R}^d$ or $\mathbb{M}^d$

In this paper, a domain stands for a *nonempty open* set. The following theorem shows that for certain unbounded domain $D \subset \mathbb{R}^d$, such as a cone or a union of quadrants, a radial function that is PD on D must be PD on $\mathbb{R}^d$.

Theorem 2.1. *For a given domain $D \subset \mathbb{R}^d$, if for any $r > 0$, there exists $\mathbf{c} \in \mathbb{R}^d$ such that the ball $\mathbb{B}_r^d(\mathbf{c}) \subset D$, a radial function that is PD on D must be PD on $\mathbb{R}^d$.*

The following theorem shows that if a continuous radial function $f(\cdot)$ on $[0, \pi]$ is PD on a domain $D \subset \mathbb{M}^d$ and has finitely many nonzero spectral coefficients, it must be PD on $\mathbb{M}^d$. In contrast, the theorem does not hold if D is an arbitrary subset of $\mathbb{M}^d$, e.g., $D = \mathbb{S}^{d-1} \subset \mathbb{S}^d$ and $f(\theta) = p_l^{((d-3)/2,(d-3)/2)}(\cos\theta)$ as shown in Lu et al. (2020). The theorem relies on the following lemma.

Lemma 2.2. *Let D be a domain in $\mathbb{M}^d$. The eigenfunctions of the Laplace–Beltrami operator Δ on $\mathbb{M}^d$ restricted to D are linearly independent.*

Theorem 2.3. *Let D be a domain in $\mathbb{M}^d$. If a continuous radial function $f(\cdot)$ on $[0, \pi]$ is PD on D and has finitely many nonzero spectral coefficients, it is PD on $\mathbb{M}^d$.*

The next theorem shows that for most domains $D \subset \mathbb{M}^d$, a continuous radial function on $[0, \pi]$ that is PD on D is not necessarily PD on $\mathbb{M}^d$. Denote the closure of D by $\bar{D}$.

Lemma 2.4. *Let D be a domain in $\mathbb{M}^d$. If $\bar{D} = \mathbb{M}^d$, for any set of N points $\{\mathbf{x}_n\}_1^N$ in $\mathbb{M}^d$, there is an isometry $\phi(\cdot)$ on $\mathbb{M}^d$ such that $\{\phi(\mathbf{x}_n)\}_1^N \subset D$.*

Lemma 2.5. *For real numbers a, $\{b_i\}_1^\infty$, and $\{t_i\}_1^\infty$,*

$$\left(a + \sum_{i=1}^{\infty} b_i t_i\right)^2 + \sum_{i=1}^{\infty} |b_i| t_i^2 \geq a^2 \left(1 + \sum_{i=1}^{\infty} |b_i|\right)^{-1},$$

provided that all series in the equation converge.

Theorem 2.6. *Let D be a domain in $\mathbb{M}^d$. If $\bar{D} = \mathbb{M}^d$, any continuous radial function on $[0, \pi]$ that is PD on D must be PD on $\mathbb{M}^d$. If $\bar{D} \neq \mathbb{M}^d$, for any $L \in \mathbb{N}_0$, there exists a continuous radial function on $[0, \pi]$ that is PD on D but has negative spectral coefficients c_l for $0 \leq l \leq L$.*

3. PD functions on the embedded ball

By Theorem 2.6, a continuous radial function that is PD on the embedded ball $\mathbb{S}^d_+$ is not necessarily PD on $\mathbb{S}^d$. Devinatz (1959) constructed radial functions that are PD on $\mathbb{S}^1_+$ but not PD on $\mathbb{S}^1$, e.g., $f(\theta) = \cos(k\theta)$ with a non-integer k. For any $d \in \mathbb{N}$, we will construct explicitly radial functions that are PD on $\mathbb{S}^d_+$ but not PD on $\mathbb{S}^d$. The next theorem, which was proved in Buhmann and Xu (2024), is included for completeness.

Theorem 3.1. *If a continuous radial function $f(\cdot)$ on $[0, \pi]$ is PD on $\mathbb{S}^d_+$, and the indices of its nonzero spectral coefficients are all odd or all even, $f(\cdot)$ is PD on $\mathbb{S}^d$.*

It should be emphasized that $f(\cdot)$ being PD on $\mathbb{S}^d_+$ does not imply that the even and odd parts of $f(\cdot)$, namely, $(f(\cdot) \pm f(\pi - \cdot))/2$, are PD on $\mathbb{S}^d_+$, so $f(\cdot)$ is not necessarily PD on $\mathbb{S}^d$. By Theorem 2.3 and Theorem 3.1, for a continuous radial function on $[0, \pi]$ that is PD on $\mathbb{S}^d_+$ but not PD on $\mathbb{S}^d$, the set of the indices of its nonzero spectral coefficients must be infinite and contains both even and odd numbers. To construct such a function explicitly, we start with two lemmas about ultraspherical polynomials $p_l^{(\alpha)}(\cdot) \equiv p_l^{(\alpha,\alpha)}(\cdot)$.

Lemma 3.2. *For $k \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, $d \in \mathbb{N}$, and $\alpha = (d-2)/2$,*

$$\int_0^1 x^{2k} p_{2n+1}^{(\alpha)}(x)(1-x^2)^\alpha dx = \frac{(-1)^n}{2} \frac{\Gamma(\alpha+1)\Gamma(k+1)\Gamma(n-k+\frac{1}{2})}{\Gamma(-k+\frac{1}{2})\Gamma(n+k+\alpha+2)}, \tag{3.1}$$

$$\int_0^1 x^{2k+1} p_{2n}^{(\alpha)}(x)(1-x^2)^\alpha dx = \frac{(-1)^n}{2} \frac{\Gamma(\alpha+1)\Gamma(k+1)\Gamma(n-k-\frac{1}{2})}{\Gamma(-k-\frac{1}{2})\Gamma(n+k+\alpha+2)}. \tag{3.2}$$

Lemma 3.3. *For $d \in \mathbb{N}$, any integer $k \geq \alpha/2 = (d-2)/4$ and $x \in [0, 1]$,*

$$x^{2k+1} = \sum_{n=0}^{\infty} c_n p_{2n}^{(\alpha)}(x), \tag{3.3}$$

where

$$c_n = \frac{(-1)^n}{2^{2\alpha+1}} \frac{\Gamma(k+1)\Gamma(n-k-\frac{1}{2})}{\Gamma(-k-\frac{1}{2})\Gamma(n+k+\alpha+2)} \frac{(4n+2\alpha+1)\Gamma(2n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(2n+1)}. \tag{3.4}$$

The convergence is absolute and uniform.

The following theorem gives a family of continuous radial functions that are PD on $\mathbb{S}^d_+$ but not PD on $\mathbb{S}^d$. They agree with the requirements imposed by Theorem 2.3 and Theorem 3.1.

Theorem 3.4. *For $d \in \mathbb{N}$ and any integer $k \geq \alpha/2 = (d-2)/4$, the isotropic kernel*

$$C(\mathbf{x}, \mathbf{y}) = -\epsilon + (\mathbf{x} \cdot \mathbf{y})^{2k+1} + \sum_{n=1}^{\infty} |c_n| p_{2n}^{(\alpha)}(\mathbf{x} \cdot \mathbf{y}), \tag{3.5}$$

where c_n is given in Eq. (3.4) and

$$\epsilon = \frac{c_0^2}{1 + \sum_{n=1}^{\infty} |c_n|},$$

is PD on $\mathbb{S}^d_+$ but not on $\mathbb{S}^d$.

4. Proofs

Theorem 2.1

Proof. A finite set of points $\{\mathbf{x}_n\}_1^N$ in $\mathbb{R}^d$ is in the ball $\mathbb{B}_r^d(\mathbf{0})$ with $r = \max_{1\le n\le N} \|\mathbf{x}_n\|$. Since there exists $\mathbf{c} \in \mathbb{R}^d$ such that $\mathbb{B}_r^d(\mathbf{c}) \subset D$, the set $\{\mathbf{x}_n + \mathbf{c}\}_1^N$ is in D. For a radial function that is PD on D, since $\rho(\mathbf{x}_i, \mathbf{x}_j) = \rho(\mathbf{x}_i + \mathbf{c}, \mathbf{x}_j + \mathbf{c})$, Eq. (1.1) holds for any set of N real numbers $\{a_n\}_1^N$. □

Lemma 2.2

Proof. Write $\mathbf{y} \in \mathbb{S}^d$ in the local coordinates $\mathbf{y} = (\sqrt{1-\|\mathbf{x}\|^2}, \mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^d$ with $\|\mathbf{x}\| < 1$. From the geodesic distance $\rho(\mathbf{y}_1, \mathbf{y}_2) = \arccos(\mathbf{y}_1 \cdot \mathbf{y}_2)$, we obtain the Riemannian metric,

$$ds^2 = \|d\mathbf{x}\|^2 + \frac{(\mathbf{x}\cdot d\mathbf{x})^2}{1-\|\mathbf{x}\|^2}.$$

On projective spaces $\mathbb{P}^d(\mathbb{R})$, $\mathbb{P}^d(\mathbb{C})$, $\mathbb{P}^d(\mathbb{H})$, $\mathbb{P}^{16}(\mathbb{O})$, write $\mathbf{y} = (1, \mathbf{z})$ in the local affine coordinates where $\mathbf{z}$ is a vector in $\mathbb{R}^d$, $\mathbb{C}^{d/2}$, $\mathbb{H}^{d/4}$, $\mathbb{O}^2$, respectively. With the geodesic length normalized as π, the geodesic distance is given by the Fubini–Study metric (Fubini, 1904), $\rho(\mathbf{y}_1, \mathbf{y}_2) = \arccos(|\mathbf{y}_1 \cdot \overline{\mathbf{y}}_2|/\|\mathbf{y}_1\|/\|\mathbf{y}_2\|)$, where $\overline{\mathbf{y}}$ is the conjugate of $\mathbf{y}$, and the metric is

$$ds^2 = \frac{d\mathbf{z}\cdot\overline{d\mathbf{z}}}{1+\mathbf{z}\cdot\overline{\mathbf{z}}} - \frac{(d\mathbf{z}\cdot\overline{\mathbf{z}})(\mathbf{z}\cdot\overline{d\mathbf{z}})}{(1+\mathbf{z}\cdot\overline{\mathbf{z}})^2}.$$

Although the Fubini–Study metric was originally for complex projective spaces, it also works for quaternionic and octonionic projective spaces. Writing the metric as $ds^2 = g_{ij}dx^i dx^j$ in the Einstein notation, the Laplace–Beltrami operator Δ is given by the well known formula in local coordinates,

$$\Delta f = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f),$$

where $|g| = |\det(g_{ij})|$ and g^{ij} are the components of the inverse matrix of g_{ij}. Since the metric g is analytic in local coordinates, Δ is an elliptic operator with analytic coefficients. By the classical result of Morrey (1958), the eigenfunctions of Δ are analytic in local coordinates.

Any linear combination of eigenfunctions of Δ, denoted by $f(\cdot)$, is analytic in the local coordinates. By the identity theorem in $\mathbb{R}^d$ (see Mityagin (2020)), since $f(\cdot)$ is analytic, the zeros of $f(\cdot)$, $\{\mathbf{x} \in \mathbb{R}^d | f(\mathbf{x}) = 0\}$, have measure zero unless $f(\cdot)$ is zero everywhere. Since $D \subset \mathbb{M}^d$ contains a nonempty open set, D has positive measure. If $f(\cdot)$ vanishes on D, $f(\cdot) \equiv 0$ on $\mathbb{M}^d$. However, since the eigenfunctions of Δ are orthogonal on $\mathbb{M}^d$, the coefficients of the linear combination for $f(\cdot)$ must be all zero. Therefore, the eigenfunctions of Δ restricted to D are linearly independent. □

Theorem 2.3

Proof. For the given radial function $f(\cdot)$, there exists $L \in \mathbb{N}$ such that for $\mathbf{x}, \mathbf{y} \in \mathbb{M}^d$,

$$f(\rho(\mathbf{x},\mathbf{y})) = \sum_{l=0}^{L} c_l p_l^{(\alpha,\beta)}(\cos\rho(\mathbf{x},\mathbf{y})).$$

Let $\mathbf{v}(\mathbf{x}) = (S_{lm}(\mathbf{x}))$ be a vector in $\mathbb{R}^J$ where $0 \le l \le L$, $1 \le m \le \dim H_l$, and $J = \sum_{l=0}^{L} \dim H_l$. By Lemma 2.2, the functions $\{S_{lm}(\cdot) | 0 \le l \le L, 1 \le m \le \dim H_l\}$ restricted to D are linearly independent. In other words, $\{\mathbf{v}(\mathbf{x}) | \mathbf{x} \in D\}$ spans $\mathbb{R}^J$. Therefore, there exist J points in D, denoted by $\{\mathbf{x}_j\}_1^J$, such that $(\mathbf{v}(\mathbf{x}_j))_{1\le j\le J}$ forms a nonsingular $J\times J$ matrix. As a result, for each $0 \le k \le L$, there exist J real numbers, denoted by $\{a_j^k\}_1^J$, such that

$$\sum_{j=1}^{J} S_{lm}(\mathbf{x}_j)a_j^k = \delta_{lk}, \quad 0 \le l \le L, \quad 1 \le m \le \dim H_l.$$

Since $f(\cdot)$ is PD on D, by the addition theorem on $\mathbb{M}^d$ given in Eq. (1.2),

$$0 \le \sum_{i=1}^{J}\sum_{j=1}^{J} a_i^k a_j^k f(\rho(\mathbf{x}_i, \mathbf{x}_j)) = \sum_{l=0}^{N}\sum_{m=1}^{\dim H_l} \frac{c_l \omega_d}{\dim H_l}\delta_{lk}^2 = c_k\omega_d,$$

which implies that $c_k \ge 0$ for $0 \le k \le L$, hence $f(\cdot)$ is PD on $\mathbb{M}^d$ by Gangolli (1967). □

Remark. The invertibility of the $J \times J$ matrix in the proof above fails if $f(\cdot)$ has infinitely many nonzero spectral coefficients. Let $\mathbf{v}(\mathbf{x}) = (S_{lm}(\mathbf{x}))$ with $l \in \mathbb{N}_0, 1 \le m \le \dim H_l$ be a vector of infinite length. By Lemma 2.2, the functions $\{S_{lm}(\cdot) | l \in \mathbb{N}_0, 1 \le m \le \dim H_l\}$ restricted to D are linearly independent. Therefore, there exists an infinite sequence of points in D, denoted by $\{\mathbf{x}_j\}$, such that the infinite set $\{\mathbf{v}(\mathbf{x}_j)\}$ is linearly independent. However, it does not imply that $\{\mathbf{v}(\mathbf{x}_j)\}$ spans the vector space of infinite sequences. For example, for $j \in \mathbb{N}$, let $\mathbf{v}(\mathbf{x}_1)_j = 1$ and $\mathbf{v}(\mathbf{x}_i)_j = \delta_{ij}$ for $i \ge 2$. The set $\{\mathbf{v}(\mathbf{x}_j)\}$ is linearly independent, but the vector $e_1 = (1, 0, 0, \ldots)$ is not in the span of $\{\mathbf{v}(\mathbf{x}_j)\}$.

Lemma 2.4

Proof. We prove by induction. Since D is nonempty, $\exists \mathbf{x} \in D$. For $N = 1$, since the isometry group of $\mathbb{M}^d$ acts on $\mathbb{M}^d$ transitively, there exists an isometry that maps $\mathbf{x}_1$ to $\mathbf{x} \in D$. Assume that the conclusion holds for N points. For a set of $N+1$ points $\{\mathbf{x}_n\}_1^{N+1}$ in $\mathbb{M}^d$, by the induction assumption, there is an isometry $\phi(\cdot)$ on $\mathbb{M}^d$ such that $\{\phi(\mathbf{x}_n)\}_1^N \subset D$. If $\phi(\mathbf{x}_{N+1}) \in D$, $\{\phi(\mathbf{x}_n)\}_1^{N+1} \subset D$. Assume that $\phi(\mathbf{x}_{N+1}) \notin D$. Since D is open, $\exists \epsilon > 0$ such that $B_\epsilon(\phi(\mathbf{x}_n)) \subset D$ for $1 \le n \le N$, where $B_\epsilon(\mathbf{c}) = \{\mathbf{x} \in \mathbb{M}^d, \rho(\mathbf{x}, \mathbf{c}) \le \epsilon\}$. Since $\bar{D} = \mathbb{M}^d$, $\exists \mathbf{x} \in B_\epsilon(\phi(\mathbf{x}_{N+1})) \cap D$. The connected component of the isometry group of $\mathbb{M}^d$ is a simple Lie group, so $\forall \epsilon > 0$ there exists an isometry $\psi(\cdot)$ on $\mathbb{M}^d$ for which $\rho(\psi(\mathbf{x}), \mathbf{x}) < \epsilon$ for all $\mathbf{x} \in \mathbb{M}^d$. In particular, there exists an isometry $\psi(\cdot)$ such that $\psi(\phi(\mathbf{x}_{N+1})) = \mathbf{x}$ and $\rho(\psi(\phi(\mathbf{x}_n)), \phi(\mathbf{x}_n)) < \epsilon$ for $1 \le n \le N$. As a result, $\{\psi(\phi(\mathbf{x}_n))\}_1^{N+1} \subset D$. □

Lemma 2.5

Proof. The left hand side is a convex function whose minimum is attained at the unique critical point where $t_i = -a \cdot \mathrm{sign}(b_i)/(1 + \sum_{i=1}^\infty |b_i|)$. □

Theorem 2.6

Proof. Assume that $\bar{D} = \mathbb{M}^d$. For any set of N points $\{\mathbf{x}_n\}_1^N$ in $\mathbb{M}^d$ and N real numbers $\{a_n\}_1^N$, by Lemma 2.4 there exists an isometry $\phi(\cdot)$ on $\mathbb{M}^d$ such that $\{\phi(\mathbf{x}_n)\}_1^N \subset D$. For a continuous radial function $f(\cdot)$ on $[0, \pi]$ that is PD on D, using Eq. (1.1),

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j f(\rho(\mathbf{x}_i, \mathbf{x}_j)) = \sum_{i=1}^N \sum_{j=1}^N a_i a_j f(\rho(\phi(\mathbf{x}_i), \phi(\mathbf{x}_j))) \ge 0,$$

therefore $f(\cdot)$ is PD on $\mathbb{M}^d$.

Next, assume that $\bar{D} \ne \mathbb{M}^d$, i.e., $\mathbb{M}^d - D$ has a nonempty open subset. Let $\eta(\cdot)$ be a bump function on $\mathbb{M}^d$ with support in $\mathbb{M}^d - D$, i.e., a nonnegative smooth function on $\mathbb{M}^d$ that is zero in D and positive in a nonempty open subset of $\mathbb{M}^d - D$. Define a kernel on $\mathbb{M}^d$,

$$g(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^L \frac{\dim H_l}{\omega_d} p_l^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{y}))\eta(\mathbf{x})\eta(\mathbf{y}).$$

Since $\eta(\cdot)$ vanishes on D, $g(\mathbf{x}, \mathbf{y}) = 0$ on D. By the addition theorem on $\mathbb{M}^d$ given in Eq. (1.2),

$$g(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^L \sum_{m=1}^{\dim H_l} \phi_{lm}(\mathbf{x})\phi_{lm}(\mathbf{y}),$$

where

$$\phi_{lm}(\mathbf{x}) = S_{lm}(\mathbf{x})\eta(\mathbf{x}).$$

Expand $\phi_{lm}(\cdot)$ in spherical harmonics on $\mathbb{M}^d$,

$$\phi_{lm}(\mathbf{x}) = \sum_{l'=0}^\infty \sum_{m'=1}^{\dim H_{l'}} a_{lm}^{l'm'} S_{l'm'}(\mathbf{x}),$$

where

$$a_{lm}^{l'm'} = \int_{\mathbb{M}^d} \phi_{lm}(\mathbf{x}) S_{l'm'}(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{M}^d} S_{lm}(\mathbf{x})\eta(\mathbf{x}) S_{l'm'}(\mathbf{x}) d\mathbf{x}.$$

Since $\phi_{lm}(\cdot)$ is a smooth function on $\mathbb{M}^d$, for any $n \in \mathbb{N}$,

$$\int_{\mathbb{M}^d} \phi_{lm}(\mathbf{x}) S_{l'm'}(\mathbf{x}) d\mathbf{x} = \frac{1}{\lambda_{l'}^n} \int_{\mathbb{M}^d} \phi_{lm}(\mathbf{x})(-\Delta)^n S_{l'm'}(\mathbf{x}) d\mathbf{x} = \frac{1}{\lambda_{l'}^n} \int_{\mathbb{M}^d} (-\Delta)^n \phi_{lm}(\mathbf{x}) S_{l'm'}(\mathbf{x}) d\mathbf{x}.$$

By the Cauchy–Schwarz inequality,

$$(a_{lm}^{l'm'})^2 \le \frac{1}{\lambda_{l'}^{2n}} \int_{\mathbb{M}^d} ((-\Delta)^n \phi_{lm}(\mathbf{x}))^2 d\mathbf{x} = \frac{1}{(l'(l' + \alpha + \beta + 1))^{2n}} \int_{\mathbb{M}^d} ((-\Delta)^n \phi_{lm}(\mathbf{x}))^2 d\mathbf{x}.$$

For $0 \le l \le L$, $1 \le m \le H_l$, and $n \in \mathbb{N}$, $|a_{lm}^{l'm'}| \le C_n (l')^{-n}$, where

$$C_n = \sqrt{\max_{0 \le l \le L, 1 \le m \le H_l} \int_{\mathbb{M}^d} ((-\Delta)^n \phi_{lm}(\mathbf{x}))^2 d\mathbf{x}}.$$

Setting $n = d + 1$ and $C = C_{d+1}$, we obtain

$$|a_{lm}^{l'm'}| \le C(l')^{-d-1}. \tag{4.1}$$

By Eq. (1.2), $|S_{lm}(\mathbf{x})| \leq \sqrt{\dim H_l/\omega_d} = O(l^{(d-1)/2})$, so the series for $\phi_{lm}(\cdot)$ converges uniformly. Let $J = \sum_{l=0}^{L} \dim H_l$, A be a $J \times J$ matrix with entries $a_{lm}^{l'm'}$ for $0 \leq l \leq L$, $1 \leq m \leq \dim H_l$, $0 \leq l' \leq L$, and $1 \leq m' \leq \dim H_{l'}$, and

$$h(\mathbf{x},\mathbf{y}) = JC \sum_{l=L+1}^{\infty} l^{-d-1} \frac{\dim H_l}{\omega_d} p_l^{(\alpha,\beta)}(\cos\rho(\mathbf{x},\mathbf{y})),$$

which converges uniformly to a continuous function because $\dim H_l = O(l^{d-1})$. For any set of N points $\{\mathbf{x}_j\}_1^N$ on $\mathbb{S}^d$ and N real numbers $\{a_j\}_1^N$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j g(\mathbf{x}_i,\mathbf{x}_j) = \sum_{l=0}^{L}\sum_{m=1}^{\dim H_l}\left(\sum_{j=1}^{N} a_j \phi_{lm}(\mathbf{x}_j)\right)^2 = \sum_{l=0}^{L}\sum_{m=1}^{\dim H_l}\left(\sum_{l'=0}^{\infty}\sum_{m'=1}^{\dim H_{l'}} a_{lm}^{l'm'} t_{l'm'}\right)^2, \tag{4.2}$$

where $t_{lm} = \sum_{j=1}^{N} a_j S_{lm}(\mathbf{x}_j)$. By the addition theorem in Eq. (1.2) and Eq. (4.1),

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j h(\mathbf{x}_i,\mathbf{x}_j) = J\sum_{l=L+1}^{\infty}\sum_{m=1}^{\dim H_l} C l^{-d-1} t_{lm}^2 \geq \sum_{l=0}^{L}\sum_{m=1}^{\dim H_l}\sum_{l'=L+1}^{\infty}\sum_{m'=1}^{\dim H_{l'}} |a_{lm}^{l'm'}| t_{l'm'}^2. \tag{4.3}$$

By Lemma 2.5,

$$\left(\sum_{l'=0}^{\infty}\sum_{m'=1}^{\dim H_{l'}} a_{lm}^{l'm'} t_{l'm'}\right)^2 + \sum_{l'=L+1}^{\infty}\sum_{m'=1}^{\dim H_{l'}} |a_{lm}^{l'm'}| t_{l'm'}^2 \geq K_{lm}^{-1}\left(\sum_{l'=0}^{L}\sum_{m'=1}^{\dim H_{l'}} a_{lm}^{l'm'} t_{l'm'}\right)^2,$$

where

$$K_{lm} = 1 + \sum_{l'=L+1}^{\infty}\sum_{m'=1}^{\dim H_{l'}} |a_{lm}^{l'm'}| \leq K \equiv 1 + C\sum_{l'=L+1}^{\infty} \dim H_{l'} (l')^{-d-1}.$$

Since $\dim H_l = O(l^{d-1})$, K is finite. Adding Eqs. (4.2) and (4.3), we have

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j (g(\mathbf{x}_i,\mathbf{x}_j) + h(\mathbf{x}_i,\mathbf{x}_j)) \geq \frac{1}{K}\sum_{l=0}^{L}\sum_{m=1}^{\dim H_l}\left(\sum_{l'=0}^{L}\sum_{m'=1}^{\dim H_{l'}} a_{lm}^{l'm'} t_{l'm'}\right)^2 = \frac{\|At\|^2}{K},$$

where $t = (t_{lm})_{0\leq l\leq L, 1\leq m\leq \dim H_l}$ is a vector in $\mathbb{R}^J$ and A is a $J \times J$ matrix with entries $a_{lm}^{l'm'}$ for $0 \leq l \leq L$, $1 \leq m \leq \dim H_l$, $0 \leq l' \leq L$, and $1 \leq m' \leq \dim H_{l'}$. By Lemma 2.2, for a nonzero vector $t \in \mathbb{R}^J$, $\sum_{l=0}^{L}\sum_{m=1}^{\dim H_l} S_{lm}(\mathbf{x}) t_{lm}$ does not vanish on a nonempty open set in $\mathbb{M}^d$, so

$$t^T A t = \int_{\mathbb{M}^d}\left(\sum_{l=0}^{L}\sum_{m=1}^{\dim H_l} S_{lm}(\mathbf{x}) t_{lm}\right)^2 \eta(\mathbf{x}) d\mathbf{x} > 0,$$

which proves that A is a positive definite matrix. Therefore, there exists $\lambda > 0$ such that $\|At\| \geq \lambda \|t\|$ for any $t \in \mathbb{R}^J$. As a result,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j (g(\mathbf{x}_i,\mathbf{x}_j) + h(\mathbf{x}_i,\mathbf{x}_j)) \geq \frac{\lambda^2}{K}\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \sum_{l=0}^{L}\frac{\dim H_l}{\omega_d} p_l^{(\alpha,\beta)}(\cos\rho(\mathbf{x}_i,\mathbf{x}_j)).$$

In other words,

$$g(\mathbf{x},\mathbf{y}) + h(\mathbf{x},\mathbf{y}) - \frac{\lambda^2}{K}\sum_{l=0}^{L}\frac{\dim H_l}{\omega_d} p_l^{(\alpha,\beta)}(\cos\rho(\mathbf{x},\mathbf{y})),$$

is PD on $\mathbb{M}^d$. Since $g(\mathbf{x},\mathbf{y}) = 0$ on D, the isotropic kernel

$$f(\mathbf{x},\mathbf{y}) \equiv h(\mathbf{x},\mathbf{y}) - \frac{\lambda^2}{K}\sum_{l=0}^{L}\frac{\dim H_l}{\omega_d} p_l^{(\alpha,\beta)}(\cos\rho(\mathbf{x},\mathbf{y}))$$

is PD on D. By the definition of $h(\cdot,\cdot)$, the spectral coefficients c_l of the radial part of $f(\cdot,\cdot)$ are positive for $l > L$ and negative for $0 \leq l \leq L$. □

Theorem 3.1

Proof. Define $g : [-1,1] \to \mathbb{R}$ by $g(z) = f(\arccos(z))$. Since the ultraspherical polynomial $p_l^{(\alpha)}(\cdot)$ has the same parity as l, the given condition is equivalent to that $g(\cdot)$ is an odd or even function. Assume $g(\cdot)$ is an odd function. For any set of points $\{\mathbf{x}_j\}_1^J$ in $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and numbers $\{a_j\}_1^J$, define $\mathbf{y}_j = -\mathbf{x}_j$ and $b_j = -a_j$ if $(\mathbf{x}_j)_1 < 0$, and $\mathbf{y}_j = \mathbf{x}_j$ and $b_j = a_j$ otherwise. Then $\{\mathbf{y}_j\}_1^J$ are in $\mathbb{S}_+^d$, and since $f(\cdot)$ is PD on $\mathbb{S}_+^d$,

$$\sum_{i=1}^{J}\sum_{j=1}^{J} a_i a_j g(\mathbf{x}_i \cdot \mathbf{x}_j) = \sum_{i=1}^{J}\sum_{j=1}^{J} b_i b_j g(\mathbf{y}_i \cdot \mathbf{y}_j) \geq 0.$$

Consequently, $f(\cdot)$ is PD on $\mathbb{S}^d$. If $g(\cdot)$ is an even function, the proof holds if we set $b_j = a_j$ for all j. □

Lemma 3.2

Proof. By the recurrence relation 18.9.1 in Olver et al.,

$$(2l+2\alpha+1)xp_l^{(\alpha)}(x) = (l+2\alpha+1)p_{l+1}^{(\alpha)}(x) + lp_{l-1}^{(\alpha)}(x), \tag{4.4}$$

we have

$$p_{2n}^{(\alpha)}(0) = (-1)^n \frac{\Gamma(\alpha+1)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+\alpha+1)}.$$

Combined with the derivative formula 18.9.20 in Olver et al., we get

$$\int_0^1 p_{2n+1}^{(\alpha)}(x)(1-x^2)^\alpha dx = -\left.\frac{(1-x^2)^{\alpha+1}p_{2n}^{(\alpha+1)}(x)}{2(\alpha+1)}\right|_0^1 = \frac{(-1)^n}{2}\frac{\Gamma(\alpha+1)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+\alpha+2)},$$

and

$$\begin{aligned}\int_0^1 xp_{2n}^{(\alpha)}(x)(1-x^2)^\alpha dx &= \int_0^1 \frac{(2n+2\alpha+1)p_{2n+1}^{(\alpha)}(x)+2np_{2n-1}^{(\alpha)}(x)}{4n+2\alpha+1}(1-x^2)^\alpha dx \\ &= \frac{(-1)^n}{2}\frac{\Gamma(\alpha+1)\Gamma(n-\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(n+\alpha+2)}.\end{aligned}$$

Therefore, Eqs. (3.1) and (3.2) hold for $k=0$. By Eq. (4.4),

$$x^{2k}p_{2n+1}^{(\alpha)}(x) = \frac{2n+2\alpha+2}{4n+2\alpha+3}x^{2k-1}p_{2n+2}^{(\alpha)}(x) + \frac{2n+1}{4n+2\alpha+3}x^{2k-1}p_{2n}^{(\alpha)}(x),$$

$$x^{2k+1}p_{2n}^{(\alpha)}(x) = \frac{2n+2\alpha+1}{4n+2\alpha+1}x^{2k}p_{2n+1}^{(\alpha)}(x) + \frac{2n}{4n+2\alpha+1}x^{2k}p_{2n-1}^{(\alpha)}(x).$$

Eqs. (3.1) and (3.2) can be proven by induction on k. □

Lemma 3.3

Proof. Since the Jacobi polynomials of even order, $p_{2n}^{(\alpha)}$ with $n \in \mathbb{N}_0$, span the space of even L_2 functions on $[-1,1]$, they form an orthogonal basis of the space of L_2 functions on $[0,1]$. By 18.3 in Olver et al.,

$$\int_0^1 p_{2n}^{(\alpha)}(x)p_{2n'}^{(\alpha)}(x)(1-x^2)^\alpha dx = \frac{2^{2\alpha}}{4n+2\alpha+1}\frac{\Gamma^2(\alpha+1)\Gamma(2n+1)}{\Gamma(2n+2\alpha+1)}\delta_{nn'}. \tag{4.5}$$

For $k \in \mathbb{N}_0$ and $x \in [0,1]$, we have the expansion

$$x^{2k+1} = \sum_{n=0}^\infty c_n p_{2n}^{(\alpha)}(x),$$

which converges in L_2 norm, and by Eq. (3.2) and Eq. (4.5),

$$\begin{aligned}c_n &= \frac{\int_0^1 x^{2k+1}p_{2n}^{(\alpha)}(x)(1-x^2)^\alpha dx}{\int_0^1 (p_{2n}^{(\alpha)}(x))^2(1-x^2)^\alpha dx} \\ &= \frac{(-1)^n}{2^{2\alpha+1}}\frac{\Gamma(k+1)\Gamma(n-k-\frac{1}{2})}{\Gamma(-k-\frac{1}{2})\Gamma(n+k+\alpha+2)}\frac{(4n+2\alpha+1)\Gamma(2n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(2n+1)}.\end{aligned}$$

Since $|c_n| \sim n^{\alpha-2k-\frac{3}{2}}$, $\sum_{n=1}^\infty |c_n| < \infty$ for $k \geq \alpha/2$. Since $p_{2n}^{(\alpha)}(\cdot)$ is PD on $\mathbb{S}^d$, $|p_{2n}^{(\alpha)}(x)| \leq p_{2n}^{(\alpha)}(0) = 1$. Therefore, the series in Eq. (3.3) converges absolutely and uniformly. □

Theorem 3.4

Proof. By Proposition 1 in Lu et al. (2023), for any $l \in \mathbb{N}_0$,

$$p_l^{(\alpha)}(\mathbf{x}\cdot\mathbf{y}) \geq p_l^{(\alpha)}(x_1)p_l^{(\alpha)}(y_1).$$

Here $A(\cdot,\cdot) \geq B(\cdot,\cdot)$ means that $A(\cdot,\cdot) - B(\cdot,\cdot)$ is a PD kernel on $\mathbb{S}^d$. In particular, $\mathbf{x}\cdot\mathbf{y} \geq x_1y_1$. Therefore,

$$C(\mathbf{x},\mathbf{y}) \geq D(\mathbf{x},\mathbf{y}) \equiv -\epsilon + (x_1y_1)^{2k+1} + \sum_{n=1}^\infty |c_n|p_{2n}^{(\alpha)}(x_1)p_{2n}^{(\alpha)}(y_1).$$

Notice that $D(\cdot,\cdot)$ is not isotropic. For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{S}^d_+$, $x_1 \geq 0$ and $y_1 \geq 0$. By Lemma 3.3, on $\mathbb{S}^d_+$,

$$D(\mathbf{x},\mathbf{y}) = -\epsilon + \sum_{n=0}^{\infty} c_n p_{2n}^{(\alpha)}(x_1) \sum_{n=0}^{\infty} c_n p_{2n}^{(\alpha)}(y_1) + \sum_{n=1}^{\infty} |c_n| p_{2n}^{(\alpha)}(x_1) p_{2n}^{(\alpha)}(y_1),$$

with summable $\sum_{n=1}^{\infty} |c_n|$ for $k \geq \alpha/2$. For any N points $\{\mathbf{x}_i\}_1^N$ in $\mathbb{S}^d_+$, N real numbers $\{a_i\}_1^N$, and $n \in \mathbb{N}_0$, denote

$$t_n = \sum_{i=1}^{N} p_{2n}^{(\alpha)}((\mathbf{x}_i)_1) a_i.$$

By Lemma 2.5,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j D(\mathbf{x}_i,\mathbf{x}_j) = -\epsilon t_0^2 + \left(\sum_{n=0}^{\infty} c_n t_n\right)^2 + \sum_{n=1}^{\infty} |c_n| t_n^2 \geq -\epsilon t_0^2 + \frac{c_0^2 t_0^2}{1+\sum_{n=1}^{\infty}|c_n|} = 0.$$

Therefore, $D(\cdot,\cdot)$ and $C(\cdot,\cdot)$ are PD on $\mathbb{S}^d_+$. On the other hand, by Eq. (3.4),

$$c_0 = \frac{\Gamma(k+1)\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(k+\alpha+2)\Gamma(\alpha+1)} \neq 0.$$

So $\epsilon > 0$. By Schoenberg (1942), an isotropic kernel on $\mathbb{S}^d$ is PD if and only if the spectral coefficients of its radial part are nonnegative and summable. Therefore, $C(\cdot,\cdot)$ is not PD on $\mathbb{S}^d$. □

5. Conclusion

We studied continuous radial functions that are positive definite (PD) on a subset of the Euclidean spaces $\mathbb{R}^d$ or compact two-point homogeneous spaces $\mathbb{M}^d$. A radial function that is PD on $D \subset \mathbb{R}^d$ which contains balls of any size is PD on $\mathbb{R}^d$. A continuous radial function on $[0,\pi]$ that is PD on a domain $D \subset \mathbb{M}^d$ and has finitely many nonzero spectral coefficients must be PD on $\mathbb{M}^d$. On the other hand, a continuous radial function on $[0,\pi]$ that is PD on a domain $D \subset \mathbb{M}^d$ is guaranteed to be PD on $\mathbb{M}^d$ if and only if $\bar{D} = \mathbb{M}^d$. Moreover, if $\bar{D} \neq \mathbb{M}^d$, there exists a continuous radial function that is PD on $D \subset \mathbb{M}^d$ with finitely many specified negative spectral coefficients. We also explicitly constructed continuous radial functions that are PD on the unit ball embedded in $\mathbb{S}^d$ but not PD on $\mathbb{S}^d$.

Acknowledgments

The author is supported by U.S. National Science Foundation Grant DMS-2008154.

Data availability

No data was used for the research described in the article.

References

Buhmann, M., Xu, Y., 2024. Positive definite functions on a regular domain. IMA J. Numer. Anal. http://dx.doi.org/10.1093/imanum/drae074.
Devinatz, A., 1959. On the extensions of positive definite functions. Acta Math. 102, 109–134.
Fubini, G., 1904. Sulle metriche definite da una forma Hermitiana. Atti del Reale Istituto Veneto di Scienze. Lett. Ed Arti 63, 501–513.
Gangolli, R., 1967. Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters. Ann. Inst. H. Poincaré Sect. B (N. S.) 3, 121–226.
Koornwinder, T., 1973. The addition formula for Jacobi polynomials and spherical harmonics. SIAM J. Appl. Math. 25 (2), 236–246.
Lu, T., Leonenko, N., Ma, C., 2020. Series representations of isotropic vector random fields on balls. Statist. Probab. Lett. 156, 108583.
Lu, T., Ma, C., Xiao, Y., 2023. Strong local nondeterminism and exact modulus of continuity for isotropic Gaussian random fields on compact two-point homogeneous spaces. J. Theor. Probab 36, 2403–2425.
Ma, C., Malyarenko, A., 2020. Time-varying isotropic vector random fields on compact two-point homogeneous spaces. J. Theor. Probab 33, 319–339.
Mityagin, B.S., 2020. The zero set of a real analytic function. Math. Notes 107, 529–530.
Morrey, Jr., C.B., 1958. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations: part I. Analyticity in the interior. Amer. J. Math. 80 (1), 198–218.
Olver, F.W.J., Daalhuis, A.B. Olde, Lozier, D.W., Schneider, B.I., Boisvert, R.F., Clark, C.W., Miller, B.R., Saunders, B.V., Cohl, H.S., McClain, M.A. (Eds.), NIST Digital Library of Mathematical Functions. https://dlmf.nist.gov/, Release 1.2.1 of 2024-06-15.
Rudin, W., 1970. An extension theorem for positive-definite functions. Duke Math. J. 37 (1), 49–53.
Schoenberg, I., 1942. Positive definite functions on spheres. Duke Math. J. 9, 96–108.
Wang, H.C., 1952. Two-point homogeneous spaces. Ann. Math. 55 (1), 177–191.