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Series representations of isotropic vector random fields on balls

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ABSTRACT

This paper deals with a class of second-order vector random fields in the unit ball of $\mathbb{R}^d$, whose direct/cross covariances are invariant or isotropic with respect to a distance defined on the ball, and gives a series representation of such an isotropic vector random field. A necessary format of covariance matrix functions is also derived for isotropic and mean square continuous vector random fields on the ball.

1. Introduction

Consider an m-variate random field $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$, where $\mathbb{B}^d$ is the unit disk or ball of radius 1 and center $\mathbf{0}$ in $\mathbb{R}^d$ ($d \geq 2$), i.e., $\mathbb{B}^d = \{\|\mathbf{x}\| \leq 1, \mathbf{x} \in \mathbb{R}^d\}$, and $\|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x} \in \mathbb{R}^d$. When $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$ has finite second-order moments, its mean function and covariance matrix function are given respectively by $\mathrm{E}\mathbf{Z}(\mathbf{x})$ and

$$\mathrm{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2)) = \mathrm{E}\{(\mathbf{Z}(\mathbf{x}_1) - \mathrm{E}\mathbf{Z}(\mathbf{x}_1))(\mathbf{Z}(\mathbf{x}_2) - \mathrm{E}\mathbf{Z}(\mathbf{x}_2))'\}, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d.$$

The objective of this paper is to investigate a class of second-order vector random fields that are isotropic on $\mathbb{B}^d$. The isotropy here is referred to all orthogonal transforms on $\mathbb{B}^d$ under a distance that differs from the usual Euclidean distance studied in Yadrenko (1983).

The distance between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ on $\mathbb{B}^d$ is defined by

$$\rho(\mathbf{x}_1, \mathbf{x}_2) = \arccos\left(\mathbf{x}_1'\mathbf{x}_2 + \sqrt{1 - \|\mathbf{x}_1\|^2}\sqrt{1 - \|\mathbf{x}_2\|^2}\right), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d, \tag{1}$$

where $\mathbf{x}_1'\mathbf{x}_2$ is the inner product in $\mathbb{R}^d$; see Bos et al. (2004), Petrushev and Xu (2008), and Dai and Xu (2013). Clearly, $0 \leq \rho(\mathbf{x}_1, \mathbf{x}_2) \leq \pi$. This distance is deduced from the geodesic distance on the hemisphere $\mathbb{S}^d_+ = \{\|\mathbf{x}\| = 1, x_{d+1} \geq 0, \mathbf{x} \in \mathbb{R}^{d+1}\}$ of $\mathbb{R}^{d+1}$ by the bijection

$$\mathbf{x} \in \mathbb{B}^d \mapsto \left(\mathbf{x}, \sqrt{1 - \|\mathbf{x}\|^2}\right)' \in \mathbb{S}^d_+,$$

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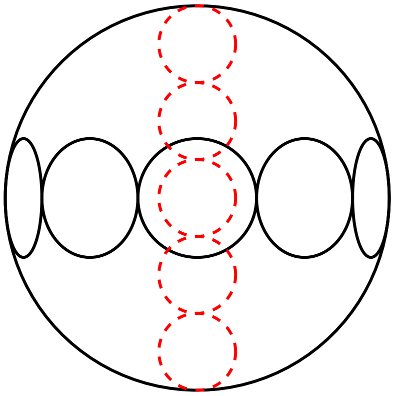

Fig. 1. Comparison of spaces with constant positive curvature and zero curvature. The solid ellipses represent balls of radius 1/5 in the spherical norm (positive curvature), while the dashed circles represent the corresponding balls in the Euclidean norm (zero curvature).

and hence it is a true distance on $\mathbb{B}^d$, where $\mathbb{S}^d = \{\|\mathbf{x}\| = 1, \mathbf{x} \in \mathbb{R}^{d+1}\}$ is a unit sphere in $\mathbb{R}^{d+1}$. It takes into account the difference between the points inside the ball and those near the boundary, in contrast to the Euclidean distance.

In cosmology, the spherical norm (1) and the Euclidean norm correspond to spaces with different curvature. In reduced-circumference polar coordinates, as discussed by Wald (1984), the metric is defined by

$$ds^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2, \tag{2}$$

where k is the constant curvature of the space. For $k = 1$, the metric is the spherical norm (1), while for $k = 0$ it is reduced to the Euclidean norm. Balls in the two norms are compared in Fig. 1.

Just like the Euclidean distance, the distance $\rho(\mathbf{x}_1, \mathbf{x}_2)$ remains the same under every orthogonal transform $\mathbf{A}$ in $\mathbb{R}^d$, *i.e.*,

$$\rho(\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2) = \rho(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d,$$

where $\mathbf{A}$ is a $d \times d$ orthogonal matrix with real entries, noticing that $\mathbf{A}\mathbf{x}_k \in \mathbb{B}^d$ whenever $\mathbf{x}_k \in \mathbb{B}^d$, $k = 1, 2$.

We call that an m-variate random field $\{\mathbf{Z}(\mathbf{x}) = (Z_1(\mathbf{x}), \ldots, Z_m(\mathbf{x}))', \mathbf{x} \in \mathbb{B}^d\}$ is isotropic, if it has second-order moments, its mean function $\mathrm{E}\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d$, does not depend on $\mathbf{x}$, and its covariance matrix function

$$\mathrm{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2)) = \mathrm{E}\{[\mathbf{Z}(\mathbf{x}_1) - \mathrm{E}\mathbf{Z}(\mathbf{x}_1)][\mathbf{Z}(\mathbf{x}_2) - \mathrm{E}\mathbf{Z}(\mathbf{x}_2)]'\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d,$$

depends only on the distance $\rho(\mathbf{x}_1, \mathbf{x}_2)$. The term isotropic random field is quite apt here, since

$$\mathrm{cov}(\mathbf{Z}(\mathbf{A}\mathbf{x}_1), \mathbf{Z}(\mathbf{A}\mathbf{x}_2)) = \mathrm{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2)), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d,$$

holds for every $d \times d$ orthogonal matrix $\mathbf{A}$. In such a case, we write $\mathbf{C}(\rho(\mathbf{x}_1, \mathbf{x}_2)) = \mathrm{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2)), \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d$, for simplicity. It is an $m \times m$ matrix function, and inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{a}_i' \mathbf{C}(\rho(\mathbf{x}_i, \mathbf{x}_j)) \mathbf{a}_j \geq 0 \tag{3}$$

holds for every $n \in \mathbb{N}$, any $\mathbf{x}_i \in \mathbb{B}^d$, and $\mathbf{a}_i \in \mathbb{R}^m$ $(i = 1, 2, \ldots, n)$, where $\mathbb{N}$ stands for the set of positive integers. On the other hand, given an $m \times m$ matrix function with these properties, there exists an m-variate Gaussian or elliptically contoured random field $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$ with $\mathbf{C}(\rho(\mathbf{x}_1, \mathbf{x}_2))$ as its covariance matrix function.

None of a theoretical or computational result can be found in the open literature about a vector random field on $\mathbb{B}^d$ that is isotropic with respect to the distance (1), which may have potential applications in many areas, such as medical imaging, atmospheric sciences, geophysics, and solar physics. Investigations of scalar and vector random fields on $\mathbb{S}^d$ isotropic with respect to the usual Euclidean distance may be found in Bingham (1973), Marinucci and Peccati (2011), Cohen and Lifshits (2012), Leonenko and Sakhno (2012), Malyarenko (2013), D'Ovidio (2014), Cheng and Xiao (2016), Ma (2016), Lu and Ma (2019) and Ma and Malyarenko (2019), among others. A series representation is provided in Section 1 for an m-variate isotropic random field on $\mathbb{B}^d$, and a necessary format is given in Section 2 of the covariance matrix function of an isotropic and mean square continuous vector random field. The proofs of theorems are in Section 3.

2. A series representation

This section presents a series representation for an m-variate isotropic random field $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$, in terms of ultraspherical or Gegenbauer's polynomials (Szegö, 1975).

For $\lambda > 0$, the ultraspherical or Gegenbauer's polynomials, $P_n^{(\lambda)}(x)$, $n \in \mathbb{N}_0$, are the coefficients of u^n in the power series expansion of the function $(1-2ux+u^2)^{-\lambda}$, i.e.,

$$(1-2ux+u^2)^{-\lambda} = \sum_{n=0}^{\infty} u^n P_n^{(\lambda)}(x), \qquad x \in \mathbb{R},\ |u| < 1, \tag{4}$$

where $\mathbb{N}_0$ stands for the set of nonnegative integers. Alternatively the ultraspherical polynomials can be defined through the recurrence formula

$$\begin{cases} P_0^{(\lambda)}(x) & \equiv & 1, \\ P_1^{(\lambda)}(x) & = & 2\lambda x, \\ P_n^{(\lambda)}(x) & = & \dfrac{2(\lambda+n-1)xP_{n-1}^{(\lambda)}(x) - (2\lambda+n-2)P_{n-2}^{(\lambda)}(x)}{n}, \qquad x \in \mathbb{R},\ n \ge 2. \end{cases}$$

They satisfy the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - (2\lambda+1)x\frac{dy}{dx} + n(2\lambda+n)y = 0,$$

and are orthonormal with respect to the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$, in the sense that

$$\int_{-1}^{1} P_i^{(\lambda)}(x)P_j^{(\lambda)}(x)(1-x^2)^{\lambda-\frac{1}{2}}dx = \begin{cases} \dfrac{\pi 2^{1-2\lambda}\Gamma(i+2\lambda)}{i!(\lambda+i)(\Gamma(\lambda))^2}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{5}$$

In the particular case $\lambda = \frac{1}{2}$, $P_n^{\left(\frac{1}{2}\right)}(x)$ $(n \in \mathbb{N}_0)$ are the Legendre polynomials. Some special cases and particular values are

$$P_n^{(1)}(\cos\vartheta) = \frac{\sin(n+1)\vartheta}{\sin\vartheta},$$
$$P_n^{(\lambda)}(1) = \binom{2\lambda+n-1}{n}.$$

For a positive-definite matrix $\mathbf{B}$, its positive-definite square root $\mathbf{B}^{\frac{1}{2}}$ is a matrix of the same order of $\mathbf{B}$ such that $\mathbf{B} = \mathbf{B}^{\frac{1}{2}}\left(\mathbf{B}^{\frac{1}{2}}\right)'$. For a sequence of $m \times m$ matrices $\{\mathbf{B}_n, n \in \mathbb{N}_0\}$, the series $\sum_{n=0}^{\infty} \mathbf{B}_n$ is said to be convergent, if each of its entries is convergent. As an example, noticing that $P_n^{\left(\frac{d-1}{2}\right)}(1) = \frac{\Gamma(n+d-1)}{\Gamma(n+1)\Gamma(d-1)} \sim \frac{n^{d-2}}{\Gamma(d-1)}$, $n \to \infty$, the convergence of $\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(1)$ is equivalent to that of $\sum_{n=0}^{\infty} n^{d-2}\mathbf{B}_n$, for $d \ge 2$. In what follows denote a positive sequence $\{\alpha_n, n \in \mathbb{N}_0\}$ by

$$\alpha_n = \left(\frac{2n+d-1}{d-1}\right)^{\frac{1}{2}}, \qquad n \in \mathbb{N}_0, \tag{6}$$

and denote by $\mathbf{I}_m$ an $m \times m$ identity matrix. A series representation is provided in the following theorem for an m-variate isotropic random field on $\mathbb{B}^d$ $(d \ge 2)$.

Theorem 1. *Suppose that* $\{\mathbf{V}_n, \in \mathbb{N}_0\}$ *is a sequence of independent m-variate random vectors with* $\mathrm{E}\mathbf{V}_n = \mathbf{0}$ *and* $\mathrm{cov}(\mathbf{V}_n, \mathbf{V}_n) = \alpha_n^2\mathbf{I}_m$, $\mathbf{U}$ *is a (d+1)-variate random vector uniformly distributed on* $\mathbb{S}^d$ *and is independent of* $\{\mathbf{V}_n, n \in \mathbb{N}_0\}$*, and that* $\{\mathbf{B}_n, n \in \mathbb{N}_0\}$ *is a sequence of m × m positive definite matrices. If the series* $\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(1)$ *converges, then*

$$\mathbf{Z}(\mathbf{x}) = \sum_{n=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}}\mathbf{V}_n P_n^{\left(\frac{d-1}{2}\right)}\left(\sum_{j=1}^{d} x_j U_j + \sqrt{1-\|\mathbf{x}\|^2}U_{d+1}\right), \qquad \mathbf{x} \in \mathbb{B}^d, \tag{7}$$

is an m-variate isotropic random field on $\mathbb{B}^d$*, its mean function is identical to* $\mathbf{0}$*, and its covariance matrix function is*

$$\mathrm{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(x_2)) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(\cos\rho(\mathbf{x}_1, \mathbf{x}_2)), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d. \tag{8}$$

The terms of (7) are uncorrelated; more precisely,

$$\mathrm{cov}\left(\mathbf{B}_i^{\frac{1}{2}}\mathbf{V}_i P_i^{\left(\frac{d-1}{2}\right)}(\cos\rho(\mathbf{x}_1, \mathbf{U})),\ \mathbf{B}_j^{\frac{1}{2}}\mathbf{V}_j P_j^{\left(\frac{d-1}{2}\right)}(\cos\rho(\mathbf{x}_2, \mathbf{U}))\right) = 0, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d,\ i \neq j.$$

Corollary 1. *For each $n \in \mathbb{N}_0$ and a positive definite matrix* $\mathbf{B}$, $\mathbf{B}P_n^{\left(\frac{d-1}{2}\right)}(\cos\rho(\mathbf{x}_1,\mathbf{x}_2))$ *is an isotropic covariance matrix function on* $\mathbb{B}^d$.

One may employ (7) to simulate an isotropic vector random field on $\mathbb{B}^d$. A similar series representation for an isotropic vector random field on $\mathbb{S}^d$ is given in Ma (2016). A helpful benefit of (8) is that it can be used to identify some isotropic covariance matrix structures on $\mathbb{B}^d$, based on those on $\mathbb{S}^d$.

Corollary 2. *Suppose that* $\mathbf{C}(x)$ *is an* $m \times m$ *continuous matrix function on* $[-1, 1]$, *and that* $(\mathbf{C}(x))' = \mathbf{C}(x), x \in [-1, 1]$. *If it makes* $\mathbf{C}(\mathbf{x}_1'\mathbf{x}_2)$ *an isotropic covariance matrix function on* $\mathbb{S}^d$, *then it also makes* $\mathbf{C}(\cos\rho(\mathbf{x}_1,\mathbf{x}_2))$ *an isotropic covariance matrix function on* $\mathbb{B}^d$.

Indeed, if $\mathbf{C}(\mathbf{x}_1'\mathbf{x}_2)$ is an isotropic covariance matrix function on $\mathbb{S}^d$, then $\mathbf{C}(x)$ is of the form

$$\mathbf{C}(x) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(x), \qquad x \in [-1, 1],$$

where $\{\mathbf{B}_n, n \in \mathbb{N}_0\}$ is a sequence of $m \times m$ positive definite matrices, and the series $\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(1)$ converges; see, for instance, Hannan (1970) and Ma (2012) . In virtue of (7), we get an m-variate isotropic random field on $\mathbb{B}^d$, whose covariance matrix function is $\mathbf{C}(\cos\rho(\mathbf{x}_1,\mathbf{x}_2))$.

Example 1. For distinct positive constants $b_1, \ldots, b_m$, the $m \times m$ matrix functions

$$C_{ij}(\cos\rho(\mathbf{x}_1,\mathbf{x}_2)) = \left(b_i^2 + b_j^2 - 2b_ib_j\cos\rho(\mathbf{x}_1,\mathbf{x}_2)\right)^{-\frac{1}{2}} \sin\left(\left(b_i^2 + b_j^2 - 2b_ib_j\cos\rho(\mathbf{x}_1,\mathbf{x}_2)\right)^{\frac{1}{2}}\right),$$
$$\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^2, i, j = 1, \ldots, m,$$

form a covariance matrix function on $\mathbb{B}^2$, due to Corollary 1 and Example 4 of Ma (2012). In this case, in terms of the Bessel function $J_{n+\frac{1}{2}}(x)$,

$$\mathbf{B}_n = \pi(n+2)\left(\frac{J_{n+\frac{1}{2}}(b_i)}{\sqrt{b_i}}\frac{J_{n+\frac{1}{2}}(b_j)}{\sqrt{b_j}}\right)_{m\times m}, \qquad n \in \mathbb{N}_0,$$

and $\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{1}{2}\right)}(1)$ converges, noticing that (see, e.g., (9) on page 366 of Watson, 1944)

$$\left(b_1^2 + b_2^2 - (2b_1b_2\cos\theta)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \sin\left(b_1^2 + b_2^2 - (2b_1b_2\cos\theta)^{\frac{1}{2}}\right) = \pi\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\frac{J_{n+\frac{1}{2}}(b_1)}{\sqrt{b_1}}\frac{J_{n+\frac{1}{2}}(b_2)}{\sqrt{b_2}}P_n^{\left(\frac{1}{2}\right)}(\cos\theta).$$

Example 2. Given constants $b_{ij} \in (-1, 1)$, $i, j = 1, \ldots, m$, if an $m \times m$ matrix $\mathbf{B}$ with entries b_{ij} is positive definite, then, by Corollary 1, the $m \times m$ matrix functions

$$C_{ij}(\cos\rho(\mathbf{x}_1,\mathbf{x}_2)) = \left(1 - 2b_{ij}\cos\rho(\mathbf{x}_1,\mathbf{x}_2) + b_{ij}^2\right)^{-\frac{d-1}{2}},$$
$$\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d, i, j = 1, \ldots, m,$$

form a covariance matrix function on $\mathbb{B}^d$, since it follows from (4) that

$$\mathbf{B}_n = (b_{ij}^n)_{m\times m}, \qquad n \in \mathbb{N},$$

and that $\sum_{n=1}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(1)$ converges.

Similarly, one may obtain another covariance matrix function on $\mathbb{B}^d$, whose entries are

$$C_{ij}(\cos\rho(\mathbf{x}_1,\mathbf{x}_2)) = (1 - b_{ij}^2)\left(1 - 2b_{ij}\cos\rho(\mathbf{x}_1,\mathbf{x}_2) + b_{ij}^2\right)^{-\frac{d}{2}},$$
$$\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d, i, j = 1, \ldots, m.$$

3. A necessary form of covariance matrix functions

It would be of great interest to derive a general form of the covariance matrix function for an m-variate isotropic and mean square continuous random field $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$. By mean square continuous, we mean that, for $k = 1, \ldots, m$,

$$\mathrm{E}|Z_k(\mathbf{x}_1) - Z_k(\mathbf{x}_2)|^2 \to 0, \quad \text{as} \quad \rho(\mathbf{x}_1,\mathbf{x}_2) \to 0, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d.$$

It implies the continuity of each entries of the associated covariance matrix function in terms of $\rho(\mathbf{x}_1, \mathbf{x}_2)$.

It is not clear what a general form could be for the covariance matrix function of an m-variate isotropic and mean square continuous random field on $\mathbb{B}^d$. Nevertheless, a necessary format is given in the following theorem.

Theorem 2. *If $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$ is an m-variate isotropic and mean square continuous random field on $\mathbb{B}^d$, then its covariance matrix function* $\text{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2))$ *is of the form*

$$\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-2}{2}\right)} (\cos\rho(\mathbf{x}_1, \mathbf{x}_2)), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d, \tag{9}$$

where $\{\mathbf{B}_n, n \in \mathbb{N}_0\}$ is a sequence of $m \times m$ positive definite matrices, and the series $\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-2}{2}\right)}(1)$ converges.

It is interesting to compare (8) and (9), which differ from each other by an index of the ultraspherical polynomials. Actually, the former is a special case of the latter. To see this, it suffices to verify that the function in Corollary 1 is of the form (9), which is done by applying an identity of L. Gegenbauer (see, for instance, (8) of Askey and Wainger, 1966),

$$P_n^{(\lambda_2)}(\cos\vartheta) = \sum_{k=0}^{\left[\frac{n}{2}\right]} a_k P_{n-2k}^{(\lambda_1)}(\cos\vartheta), \qquad \vartheta \in [0, \pi],\ n \in \mathbb{N}_0,$$

where $0 < \lambda_1 < \lambda_2$, $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$, and

$$a_k = \frac{\Gamma(\lambda_1)(n-2k+\lambda_1)\Gamma(k+\lambda_2-\lambda_1)\Gamma(n-k+\lambda_2)}{\Gamma(\lambda_2)\Gamma(\lambda_2-\lambda_1)k!\Gamma(n-k+\lambda_1+1)}, \qquad k = 0, 1, \ldots, \left[\frac{n}{2}\right].$$

As a conjecture, (9) might be also the covariance matrix function of an m-variate isotropic random field on $\mathbb{B}^d$, generally speaking.

4. Proofs

4.1. Proof of Theorem 1

Since $\left|P_n^{\left(\frac{d-1}{2}\right)}(\cos\vartheta)\right| \le P_n^{\left(\frac{d-1}{2}\right)}(1)$, $n \in \mathbb{N}_0$, the convergent assumption of the series $\sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(1)$ ensures not only the mean square convergence of the series at the right hand of (7), but also the uniform and absolute convergence of the series at the right hand side of (8). In fact, for $\mathbf{x}_k \in \mathbb{B}^d$ and $n_k \in \mathbb{N}$ $(k = 1, 2)$, we have

$$\begin{aligned}
&\mathrm{E}\left[\sum_{n=n_1}^{n_1+n_2} \mathbf{B}_n^{\frac{1}{2}} \mathbf{V}_n P_n^{\left(\frac{d-1}{2}\right)}\left(\sum_{i=1}^{d} x_{1i}U_i + \sqrt{1-\|\mathbf{x}_1\|^2}U_{d+1}\right)\right.\\
&\left.\times\left(\sum_{l=n_1}^{n_1+n_2} \mathbf{B}_l^{\frac{1}{2}} \mathbf{V}_l P_l^{\left(\frac{d-1}{2}\right)}\left(\sum_{j=1}^{d} x_{2j}U_j + \sqrt{1-\|\mathbf{x}_2\|^2}U_{d+1}\right)\right)\right]'\\
=& \sum_{n=n_1}^{n_1+n_2}\sum_{l=n_1}^{n_1+n_2} \mathbf{B}_n^{\frac{1}{2}} \mathrm{E}\left(\mathbf{V}_n\mathbf{V}_l'\right)\left(\mathbf{B}_l^{\frac{1}{2}}\right)'\\
&\times \mathrm{E}\left[P_n^{\left(\frac{d-1}{2}\right)}\left(\sum_{i=1}^{d} x_{1i}U_i + \sqrt{1-\|\mathbf{x}_1\|^2}U_{d+1}\right)P_l^{\left(\frac{d-1}{2}\right)}\left(\sum_{j=1}^{d} x_{2j}U_j + \sqrt{1-\|\mathbf{x}_2\|^2}U_{d+1}\right)\right]\\
=& \sum_{i=n_1}^{n_1+n_2} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)}(\cos\rho(\mathbf{x}_1, \mathbf{x}_2))\\
\to& \ \mathbf{0}, \qquad\qquad n_1 \to \infty, n_2 \to \infty,
\end{aligned}$$

where the first equality follows from the independent assumption between $\mathbf{U}$ and $\{\mathbf{V}_n, n \in \mathbb{N}_0\}$, and the second from Lemma 3 of Ma (2016). Thus, the series at the right hand side of (7) converges in mean square.

Notice that $\left(x_1, \ldots, x_d, \sqrt{1-\|\mathbf{x}\|^2}\right)' \in \mathbb{S}^d$, whenever $\mathbf{x} \in \mathbb{B}^d$. Under the independent assumption between $\mathbf{U}$ and $\{\mathbf{V}_n, n \in \mathbb{N}_0\}$, we obtain the mean and covariance matrix functions of $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$ from Lemma 3 of Ma (2016), with

$$\mathrm{E}\mathbf{Z}(\mathbf{x}) = \sum_{n=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}} \mathrm{E}\mathbf{V}_n \mathrm{E}P_n^{\left(\frac{d-1}{2}\right)}\left(\sum_{j=1}^{d} x_jU_j + \sqrt{1-\|\mathbf{x}\|^2}U_{d+1}\right) = \mathbf{0}, \quad \mathbf{x} \in \mathbb{B}^d,$$

and

$$
\begin{aligned}
&\operatorname{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2)) \\
&= \operatorname{cov}\left(\sum_{n=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}} \mathbf{V}_n P_n^{\left(\frac{d-1}{2}\right)} \left(\sum_{i=1}^{d} x_{1i} U_i + \sqrt{1 - \|\mathbf{x}_1\|^2} U_{d+1} \right), \right. \\
&\qquad \left. \sum_{l=0}^{\infty} \mathbf{B}_l^{\frac{1}{2}} \mathbf{V}_l P_l^{\left(\frac{d-1}{2}\right)} \left(\sum_{j=1}^{d} x_{2j} U_j + \sqrt{1 - \|\mathbf{x}_2\|^2} U_{d+1} \right) \right) \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \mathbf{B}_n^{\frac{1}{2}} \mathrm{E}(\mathbf{V}_n \mathbf{V}_l') \left(\mathbf{B}_l^{\frac{1}{2}} \right)' \\
&\quad \times \mathrm{E}\left\{ P_n^{\left(\frac{d-1}{2}\right)} \left(\sum_{i=1}^{d} x_{1i} U_i + \sqrt{1 - \|\mathbf{x}_1\|^2} U_{d+1} \right) P_l^{\left(\frac{d-1}{2}\right)} \left(\sum_{j=1}^{d} x_{2j} U_j + \sqrt{1 - \|\mathbf{x}_2\|^2} U_{d+1} \right) \right\} \\
&= \sum_{n=0}^{\infty} \mathbf{B}_n \operatorname{cov}\left(\alpha_n P_n^{\left(\frac{d-1}{2}\right)} \left(\sum_{i=1}^{d} x_{1i} U_i + \sqrt{1 - \|\mathbf{x}_1\|^2} U_{d+1} \right), \right. \\
&\qquad \left. \alpha_n P_n^{\left(\frac{d-1}{2}\right)} \left(\sum_{j=1}^{d} x_{2j} U_j + \sqrt{1 - \|\mathbf{x}_2\|^2} U_{d+1} \right) \right) \\
&= \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-1}{2}\right)} (\cos \rho(\mathbf{x}_1, \mathbf{x}_2)), \qquad \mathbf{x}_j = (x_{j1}, \ldots, x_{jd})' \in \mathbb{B}^d, \ j = 1, 2.
\end{aligned}
$$

4.2. *Proof of Theorem 2*

If $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$ is an m-variate isotropic and mean square continuous random field, then its covariance matrix function $\operatorname{cov}(\mathbf{Z}(\mathbf{x}_1), \mathbf{Z}(\mathbf{x}_2))$ is continuous with respect to $\rho(\mathbf{x}_1, \mathbf{x}_2)$ on $\mathbb{B}^d$, and is continuous with respect to $\cos(\rho(\mathbf{x}_1, \mathbf{x}_2))$ as well. According to the results in Section 9.1 of Szegö (1975), each entry of $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ can be expressed as

$$
C_{ij}(\mathbf{x}_1, \mathbf{x}_2) = \sum_{n=0}^{\infty} b_n^{(i,j)} P_n^{\left(\frac{d-2}{2}\right)} (\cos(\rho(\mathbf{x}_1, \mathbf{x}_2))), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^d, \ i, j = 1, \ldots, m.
$$

In other words, $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ has to take the form (9). In particular, $\mathbf{C}(\mathbf{x}, \mathbf{x}) = \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-2}{2}\right)}(1)$ converges.

To verify that $\mathbf{B}_n$ is positive definite for each $n \in \mathbb{N}_0$, notice that the restriction of $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{B}^d\}$ on the surface of $\mathbb{B}^d$, $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{S}^{d-1}\}$, is an m-variate isotropic random function on the unit sphere $\mathbb{S}^{d-1}$, with covariance matrix function

$$
\begin{aligned}
\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) &= \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-2}{2}\right)} (\cos(\rho(\mathbf{x}_1, \mathbf{x}_2))) \\
&= \sum_{n=0}^{\infty} \mathbf{B}_n P_n^{\left(\frac{d-2}{2}\right)} (\cos(\vartheta(\mathbf{x}_1, \mathbf{x}_2))), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^{d-1},
\end{aligned}
$$

where $\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos(\mathbf{x}_1' \mathbf{x}_2)$ is the great circle distance on the sphere $\mathbb{S}^{d-1}$. By Theorem 1 of Ma (2012), $\mathbf{B}_n$ must be a positive definite matrix, for each $n \in \mathbb{N}_0$.

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