1. Introduction

This is a short overview for the beginners (graduate students and advanced undergraduates) on some aspects of biholomorphic maps.

A mathematical theory identifies objects that are “similar in every detail” from the point of view of the corresponding theory. For instance, in algebra we use the notion of isomorphism of algebraic objects (groups, rings, vector spaces, etc.), while in topology homeomorphic topological spaces are considered similar in every detail. The idea is to classify the objects under consideration.

In complex analysis the corresponding notion is the biholomorphism of the main objects in the theory: complex manifolds. Two complex manifolds $M_1, M_2$ are biholomorphic if there is a bijective holomorphic map $F : M_1 \rightarrow M_2$. Like in those other theories one wants to find the biholomorphic classification of complex manifolds. This attempt is certainly an important motivation to study biholomorphic transformations. The classification problem appears to be a very difficult problem, and only partial results (though very important and highly interesting) are known.

The Riemann Mapping Theorem states that any two proper simply connected domains in the complex plane are conformally equivalent. This is an exception. The general case is: for any $n \geq 1$ in $\mathbb{C}^n$ any two “randomly” picked domains are not biholomorphic. We will mention three examples to support this viewpoint. We also are pointing out that in case $n = 1$ the classification problem is well researched and mostly understood. For $n \geq 2$ it is way more complicated, and by now the pursuit of it has produced many very interesting and deep results and also created useful tools for SCV. Our intention is to put together various related problems, reflecting the authors interests, and in most cases refer the reader to some published material where details can be found. There are many surveys on most topics we consider. Quick search on MathSciNet shows that there are over two thousand papers published on biholomorphic maps. By no means this article is a comprehensive review. It should be considered as a glance into some of the results in the theory of these transformations.

The approximate outline of this exposition follows.

We start with three examples of non-equivalence for some “similar” domains. Then we introduce invariant metrics. Later we mention some important biholomorphic invariants and the related question of the extension of a biholomorphic mapping to the boundary. After that we consider the automorphism group of a domain in $\mathbb{C}^n$, which is a biholomorphic invariant, and related results. In the end we introduce and analyze “approximate” biholomorphic relations.

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So we start with some examples.

2. THREE EXAMPLES

The Riemann Mapping Theorem guarantees the biholomorphism of two proper simply connected domains in \( \mathbb{C} \). But once we look at non-simply connected domains the situation drastically changes. Here’s a result for annuli.

I). (F.H. Schottky, 1877) \[52\] Two annuli in \( \mathbb{C} \): \( \Omega(r_i, R_i) = \{ r_i < |z| < R_i \} ; i = 1, 2 \) are conformally equivalent if and only if \( r_1/R_1 = r_2/R_2 \).

**Proof.** (sketch) Denote the disc of radius \( r \) with center at the origin by \( \Delta(r) \). Without any loss of generality we may assume that \( r_1 = r_2 = 1 \), \( R_1 > R_2 \), and that \( f : \Omega(1, R_1) \rightarrow \Omega(1, R_2) \) is continuous to the boundary (we’ll address the necessity of this in a later section), and maps the unit circle \( \partial \Delta(1) \) of \( \partial \Omega(1, R_1) \) into the unit circle of \( \partial \Delta(1) \subset \partial \Omega(1, R_2) \). We will use the Schwarz reflection principle. The reflection over \( \partial \Delta(1) \) extends \( f \) to an analytic map of from \( \Omega(R_1^{-1}, R_1) \) to \( \Omega(R_2^{-1}, R_2) \). We now continue to reflect over the smaller circle \( \partial \Delta(R_1^{-1}) \) and get \( f \) extended to \( \Omega(R_1^{-3}, R_1) \). Continue this process indefinitely we get \( f \) extended to a conformal map of two punctured discs \( \Omega(0, R_1) \rightarrow \Omega(0, R_2) \). Since \( f \) is bounded it can be extended to \( \{ 0 \} ; f : \overline{\Omega}(0, R_1) \rightarrow \overline{\Omega}(0, R_2) \). Since \( f \) is now holomorphic, near the origin it will satisfy \( |f(z)| \leq C|z| \) for some constant \( C \). This means that for large \( n \) we have \( |R_2^{-n}| \leq C |R_1^{-n}| \). This cannot happen since by assumption \( R_1 > R_2 \). □

For \( n > 1 \) in \( \mathbb{C}^n \), a theorem analogous to the Riemann Mapping Theorem does not hold even in the following case.

II). (Henri Poincaré, 1907) For \( n \geq 2 \) the unit ball \( B^n = \{(z_1, ..., z_n) \mid \sum_{j=1}^{n} |z_j|^2 < 1 \} \) and the polydisc \( \Delta^n = \{(z_1, ..., z_n) \mid |z_i| < 1, i = 1, ..., n \} \) are not biholomorphically equivalent.

Later on several proofs of this theorem will be provided.

In \( \mathbb{C}^n \), \( n > 1 \) small perturbations of the unit ball may create a continuum of non-biholomorphic domains.

III). (Burns-Shnider-Wells thm, 1978) We’ll present one consequence of the main theorem from [7].

Consider the unit ball \( B^n \subset \mathbb{C}^n, n \geq 2, \epsilon > 0 \). Then in the \( \epsilon \)-neighborhood of \( B^n \) there is an uncountable number of mutually non-biholomorphic domains with real analytic boundaries, containing the ball and homeomorphic to the ball.

3. SOME INVARIANT METRICS

By introducing an invariant metric into a domain in \( \mathbb{C}^n \) one can then consider every biholomorphic map from one domain to another as an isometry with all the consequences coming out of that. There are several invariant measures: Kobayashi, Caratheodory, Bergman, and others. We refer the reader to a number of books written on this. It is quite an extensive topic, and we will only touch on this subject here. It will end with a proof of Poincare’s example mentioned in the previous section.
Definition 3.1. Let $D$ be a domain in $\mathbb{C}^n$. The Carathéodory and Kobayashi infinitesimal pseudo-metrics are functions from $D \times \mathbb{C}^n$ to $[0, \infty)$ defined by

\[
C_D(z, v) = \sup \{|dg(z)(v)| : g \in \mathcal{O}(D, \Delta), g(z) = 0\},
\]
\[
K_D(z, v) = \inf \{|u| : u \in \mathbb{C}, f \in \mathcal{O}(\Delta, D), f(0) = z, df(0)(u) = v\}.
\]

The Kobayashi indicatrix of $D$ at $z$ is

\[
I_{D,z} := \{v \in \mathbb{C}^n : K_D(z, v) < 1\}.
\]

Kobayashi Extremal maps exist when $D$ is bounded.

It was proved by Royden ([48] pp. 125–137) that every Kobayashi hyperbolic complex manifold is infinitesimally Kobayashi non-degenerate. The converse is false ([37] Remark 3.5.11).

Proposition 3.2. $C_D \leq K_D$.

Proof. Let $z \in D$ and $v \in \mathbb{C}^n$. Let $\varepsilon > 0$ be given. There is an $f \in \mathcal{O}(D, \Delta)$ and $u \in \mathbb{C}^n$ such that $f(0) = z$, $df(0)(u) = v$, and $|u| < K_D(z, v) + \varepsilon$. There is a $g \in \mathcal{O}(D, \Delta)$ such that $g(z) = 0$, and $|dg(z)(v)| > C_D(z, v) - \varepsilon$. By Schwarz lemma, $|d(g \circ f)(0)(u)| \leq |u|$. It follows that

\[
C_D(z, v) - \varepsilon < |dg(z)(v)| = |d(g \circ f)(0)(u)| \leq |u| < K_D(z, v) + \varepsilon.
\]

Letting $\varepsilon \to 0$ yields that $C_D(z, v) \leq K_D(z, v)$. \hfill \Box

Theorem 3.3. The Kobayashi infinitesimal pseudo-metric is decreasing under holomorphic maps in the sense that if $f \in \mathcal{O}(D, \Omega)$ then $K_\Omega(f(z), df(z)(v)) \leq K_D(z, v)$.

Proof. Let $\varepsilon > 0$. Choose $h \in \mathcal{O}(\Delta, D)$ and $u \in \mathbb{C}$ so that $h(0) = z$, $dh(0)(u) = v$, and $K_D(z, v) + \varepsilon > |u|$. Since $f \circ h \in \mathcal{O}(\Delta, \Omega)$, and $d(f \circ h)(0)(u) = df(z)(v)$, we see that $|u| \geq K_\Omega(f(z), df(z)(v))$. Thus $K_D(z, v) + \varepsilon > K_\Omega(f(z), df(z)(v))$ for each $\varepsilon > 0$. \hfill \Box

Corollary 3.4. The Kobayashi infinitesimal pseudo-metric is invariant under biholomorphic maps in the sense that if $f$ is a biholomorphic mapping from $D$ onto $\Omega$ then

\[
K_\Omega(f(z), df(z)(v)) = K_D(z, v).
\]

Moreover $df(z)(I_{D,z}) = I_{\Omega,f(z)}$.

Theorem 3.5. Let $n > 1$. Then $\Delta^n$ and $B^n$ are not holomorphically equivalent.

Proof. Suppose that $g$ is a biholomorphic mapping from $B^n$ onto $\Delta^n$. Let $a = g(0)$. There is an $h \in \text{Aut}(\Delta^n)$ with $h(a) = 0$. Let $f = h \circ g$. Then $f(0) = 0$. By Corollary 3.4, $df(0)(B^n) = \Delta^n$. That is impossible because $B^n$ has a smooth boundary and $\Delta^n$ does not. \hfill \Box

4. Boundary Extension of a Biholomorphic Map

A set of biholomorphic invariants has been introduced by S. S. Chern, J.K. Moser [12], and N. Tanaka [53]. These invariants can be used for $C^\infty$ manifolds of co-dimension one in $\mathbb{C}^n$. To use them for the case of a biholomorphism $f : D_1 \to D_2$ of bounded domains in $\mathbb{C}^n$ with smooth boundaries one needs first to prove that $f$ can be extended smoothly to the boundary. We will now discuss this question of extending a biholomorphic map between
two bounded domains in $\mathbb{C}^n$ to a diffeomorphism or even a homeomorphism between the closures of these domains.

In the case of one complex variable the conformal map $f : D \to \Omega$ for bounded domains in $\mathbb{C}$ can be extended to a diffeomorphism $\tilde{D} \to \tilde{\Omega}$ if the boundaries $\partial D, \partial \Omega$ are smooth, and to a homeomorphism is these boundaries are piece-wise smooth simple closed curves. An examination of this problem in $\mathbb{C}$ one can find in [46].

In the case of $\mathbb{C}^n, n \geq 2$ the situation is much more complicated. A short counterexample to the extendability is given by Fridman [20]: two domains $D, \Omega$ are constructed in $\mathbb{C}^2$, both biholomorphic to the bidisk, both have piece-wise smooth boundaries, and there is a $C^\infty$ diffeomorphism between them that extends smoothly to their closures. However no biholomorphic mapping $F : D \to \Omega$, nor $F^{-1}$ can be extended continuously to the boundary. If $D$ and $\Omega$ are strictly pseudoconvex domains in $\mathbb{C}^n$ with $C^\infty$ smooth boundaries the result of N. Vormoor (and independently G. Henkin)[58, 35] shows that any biholomorphic map between them extends to a homeomorphism of their closures.

The question of smooth extendability is a very difficult problem. A major breakthrough was made by C. Fefferman in 1974. He proved the following theorem [14].

**Theorem 4.1.** Let $f$ be a biholomorphic map from a domain $D \subset \mathbb{C}^n$ to another domain $\Omega \subset \mathbb{C}^n$. Suppose that $D$ and $\Omega$ are bounded strongly pseudoconvex domains with $C^\infty$-smooth boundaries. Then $f$ extends $C^\infty$ to the boundary.

Fefferman’s original proof consists of two parts. First he considers the behavior of the Bergman kernel near the boundary, which leads to precise estimates of the Bergman metric near the boundary (for definitions and main results regarding the Bergman kernel see the book by S. Krantz [41]). In the second part he proves the theorem using those estimates and the above mentioned Vormoor’s result. We would like to note that [14] is 65 pages long, and the introduction (8 pages), has a very clear sketch of both parts of the proof.

Following this major accomplishment, there have been a number of publications refining and presenting different proofs of the theorem. The book [41] has a clear and different proof of this theorem. We should also mention two more proofs that are widely cited: those by F. Forstneric [15], and S. Bell & E. Ligocka [4].

The proof of F. Forstneric is based on two classical results: the edge-of-the-wedge theorem and the Julia-Caratheodory theorem (for these classical results see e.g. [49, 50]). In this sense it is considered elementary.

Bell and Ligocka’s proof of Fefferman’s theorem made use of the Bergman kernel. For a bounded domain $D \subset \mathbb{C}^n$, the Bergman kernel function $K(z, w)$ is defined by $K(z, w) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}$, where $\{\varphi_j\}$ is an orthonormal basis for the Bergman space $B^2 := L^2(D) \cap H(D)$. The Bergman kernel does not depend on the orthonormal basis and it has the reproducing property that for each $f \in B^2(D), f(z) = \int_D K(z, w)f(w)\,dw$.

Let $W^s_0$ denote the closure of $C_0^\infty(D)$ in the Sobolev space $W^s(D)$. Let $H^s(D) := W^s(D) \cap H(D)$ be the subspace of $W^s(D)$ consisting of holomorphic functions. A domain $D \subset \mathbb{C}^n$ is said to satisfy Codition R if it is bounded with smooth boundary and if for every $s \geq 0$ there exist constants $M > 0$ and $C > 0$ such that the Bergman projection $P : L^2(D) \to B^2(D)$ satisfies $\| Pf \|_{H^s(D)} \leq C \| f \|_{W^{s+M}_0(D)}$.

**Theorem 4.2.** (Bell and Ligocka [4]) If $D_1$ and $D_2$ satisfy Condition R, then any biholomorphic mapping between $D_1$ and $D_2$ extends smoothly to the boundary.
It had been known that subelliptic estimates for the $\bar{\partial}$-Neumann operator imply Condition R and that the $\bar{\partial}$-Neumann operator satisfies the subelliptic estimates for bounded strictly pseudoconvex domains with smooth boundary. Thus Bell and Ligocka’s theorem implies the Fefferman Theorem.

Bell and Ligocka firstly proved that Condition R implies the following two conditions:

Condition A. $K(\cdot, w) \in C^\infty(D)$ for each $w \in D$.

Condition B. For each $z_0 \in \partial D$, there are $n + 1$ points $a_0, \ldots, a_n$ in $D$ such that $K(z_0, a_j) \neq 0$ and

$$\det\left(\frac{\partial K(z_0, a_j)}{\partial z_i}(z_0, a_j)\right)_{i=1, \ldots, n, j=0, 1, \ldots, n} \neq 0.$$  

Using Conditions A and B, they proved that for a given biholomorphic map $h$ between domains $D_1$ and $D_2$ the Jacobian functions $Jh(z)$ and $(Jh(z))^{-1}$ are bounded on $D_1$. Assume that $(Jh(z))^{-1}$ is not bounded. Then there exists a sequence $z_n \rightarrow z_0 \in \partial D_1$ such that $Jh(z_n) \rightarrow 0$. The transformation rule for the Bergman kernel function

$$K_1(z_n, a) = K_2(h(z_n), h(a))Jh(z_n)Jh(a)$$

and Condition A yield that $K_1(z_0, a) = 0$ for each $a \in D_1$, which contradicts Condition B.

Consider a point $z_0 \in \partial D_1$ and a sequence $\{z_n\}$ in $D_1$ with $z_n \rightarrow z_0$ such that $h(z_n)$ tends to a point $s_0 \in \partial D_2$. By Condition B, there are points $b_0, b_1, \ldots, b_n$ such that $K(s_0, b_0) \neq 0$ and

$$Q := \det\left(\frac{\partial K_2(s_0, b_j)}{\partial s_i}(s_0, b_j)\right)_{i=1, \ldots, n, j=0, 1, \ldots, n} \neq 0.$$  

Let $v_j(s) = K_2(s, b_j)/K_2(s, b_0)$, $j = 1, \ldots, n$. Then $\det(\frac{\partial v_j}{\partial s_i}(s_0)) = K_2(s_0, b_0)^{-n-1}Q \neq 0$. Set $a_j = h^{-1}(b_j)$ and $u_j(z) = K_1(z, a_j)/K_1(z, a_0)$, $j = 1, \ldots, n$. The relation

$$\left(\frac{\partial u_j}{\partial z_i}(z_0)\right) = \left(\frac{\partial v_j}{\partial s_i}(s_0)\frac{Jh(a_j)}{Jh(a_0)}\right) \cdot \left(\frac{\partial h_j}{\partial z_i}(z_0)\right)$$

then tells us that $\frac{\partial h_j}{\partial z_i}(z_0)$ are bounded. This implies that $\frac{\partial h_j}{\partial z_i}$ are bounded on $D_1$. Therefore, $h$ is continuous up to the boundary. In particular $h(z_0) = s_0$. Since

$$\det(\frac{\partial u_j}{\partial z_i}(z_0)) \neq 0, \text{ and } \det(\frac{\partial v_j}{\partial s_i}(s_0)) \neq 0,$$

the functions $u_j$ and $v_j$ are smooth local coordinates near $z_0$ and $s_0$ respectively. With respect to these coordinates the mapping $h$ is expressed as a linear mapping given by $v_j = \frac{Jh(a_j)}{Jh(a_0)}u_j$. It follows that $h$ can be extended smoothly to the boundary in a neighborhood of $z_0$.

5. Symmetry: Automorphism group of a domain

5.1. Automorphism groups. A biholomorphic map of a manifold $M$ onto itself is called an automorphism of $M$. The set $\text{Aut}(M)$ of all automorphisms of $M$ forms a group. This group is clearly a biholomorphic invariant. There is a relatively recent survey of this group by S. Krantz [40], and we’ll refer to this paper throughout the section.
A good example of using $\text{Aut}(D)$ as invariant is Poincare’s original proof of biholomorphic non-equivalence of the unit ball $B^n$ and polydisk $\Delta^n$ (for $n > 1$) by comparison of their automorphisms groups; they happen to be Lie groups of different dimensions and therefore not isomorphic.

There has been a lot of effort devoted to the study of $\text{Aut}(D)$. As with all known invariants, this one is interesting (giving an idea of how ”symmetric” a domain is) but by no means defining (even for $n = 1$ two annuli have the same automorphism group but might not be conformally equivalent). Moreover after many deep studies one may conclude that for a general “random” domain in $\mathbb{C}^n$ the $\text{Aut}(D) = \{\text{id}\}$. However there is a large set of domains with non-trivial group, and this is a justification for detailed study of $\text{Aut}(D)$.

One of the first results was the paper by H. Cartan (1935) [10] which proved that $\text{Aut}(D)$ is a real Lie group for any bounded domain $D$ in $\mathbb{C}^n$. Bedford-Dadok and Saerens-Zame [3, 51], proved that every compact Lie group can be realized as an $\text{Aut}(D)$ for some strictly pseudoconvex domain with a smooth boundary in a suitable $\mathbb{C}^n$.

5.2. Description of automorphism groups and characterization of domains, manifolds with a given group. Below for convenience we will use two notations for the holomorphic automorphism of $D$: $\text{Aut}_D$ or $\text{Aut}(D)$.

Let $D$ be a complex manifold. Consider a fixed point $w \in D$. The automorphisms of $D$ which fix $w$ form the isotropy group at $w$,

$$\text{Aut}_{D,w} := \{\varphi \in \text{Aut}_D : \varphi(w) = w\}.$$ 

For $\varphi \in \text{Aut}_D$ the left coset of $\text{Aut}_{D,w}$ with respect to $\varphi$ is defined to be

$$\varphi \text{Aut}_{D,w} := \{\varphi g : g \in \text{Aut}_{D,w}\}.$$ 

For $\varphi, \psi \in \text{Aut}_D$, if $\psi^{-1} \varphi \notin \text{Aut}_{D,w}$ then $\varphi \text{Aut}_{D,w} \cap \psi \text{Aut}_{D,w} = \emptyset$; if $\psi^{-1} \varphi \in \text{Aut}_{D,w}$ then $\varphi \text{Aut}_{D,w} = \psi \text{Aut}_{D,w}$. The cosets space

$$\text{Aut}_D / \text{Aut}_{D,w} := \{\varphi \text{Aut}_{D,w} : \varphi \in \text{Aut}_D\}$$

is naturally a real analytic manifold. The map from the cosets space $\text{Aut}_D / \text{Aut}_{D,w}$ to the orbit $\text{Aut}_D(w) := \{g(w) : g \in \text{Aut}_D\}$ given by $\varphi \text{Aut}_{D,w} \mapsto \varphi(w)$ is a bijective, real analytic map. Thus we have the following

**Proposition 5.1.** The orbit $\text{Aut}_D(w)$ is a real analytic submanifold of $D$. Its dimension is $\dim \text{Aut}_D(w) = \dim \text{Aut}_D - \dim \text{Aut}_{D,w}$.

Let $D$ be a bounded domain in $\mathbb{C}^n$ or a Kobayashi hyperbolic manifold of complex dimension $n$. It follows from the normal family theory that $\text{Aut}_{D,w}$ is compact. If $\varphi \in \text{Aut}_{D,w}$ is an automorphism which fixes $w$, its tangent map $d\varphi_w$ is a member of $\text{GL}(n, \mathbb{C})$. When $d\varphi_w = \text{id}$, by looking at iterations of series expansions of $\varphi$ at $w$, we see that $\varphi$ must be the identity map. This implies that the map $\text{Aut}_{D,w} \to \text{GL}(n, \mathbb{C})$, $\varphi \mapsto d\varphi_w$, is injective. Thus $\text{Aut}_{D,w}$ is isomorphic to a compact subgroup of $\text{GL}(n, \mathbb{C})$. Since each compact subgroup of $\text{GL}(n, \mathbb{C})$ is conjugate to a subgroup of the unitary group $U(n)$, it follows that $\text{Aut}_{D,w}$ is isomorphic to a subgroup of $U(n)$. Thus $\dim \text{Aut}_{D,w} \leq \dim U(n) = n^2$.

A few examples of the $\text{Aut}(D)$:

1. The automorphism group of the projective space: $\text{Aut}(\mathbb{P}^n)$ is isomorphic to $PGL(n + 1, \mathbb{C})$. The proof can be found in [1], p. 41.
The detailed construction of the automorphism group of the ball $B^n$ and the polydisc $\Delta^n$ can be found in [1, p. 54]. As it is shown in that book the $\text{Aut}(B^n)$ is a real Lie group of dimension $n^2 + 2n$, and $\dim(\text{Aut}(\Delta^n)) = 3n$. So, these groups have different dimensions for $n > 1$, and therefore not isomorphic for $n \geq 2$.

Here’s a way to describe the $\text{Aut}(B^n)$.

Let $0 < \lambda < 1$ and let $f : B^n \to \mathbb{C}^n$ be defined by

\begin{align*}
 f_1(z) &= \frac{z_1 + \lambda}{1 + \lambda z_1}, \\
 f_j(z) &= \frac{\sqrt{1 - \lambda^2}}{1 + \lambda z_1} z_j, \quad j = 2, \ldots, n.
\end{align*}

Then $f \in \text{Aut}(B^n)$ and $f(0) = (\lambda, 0, \ldots, 0)$. Fix a point $a \neq 0$ in $B^n$. There is a unitary transformation $U$ such that $Ua = (|a|, 0, \ldots, 0)$. Let $h$ be the transformation in (1) with $\lambda$ replaced by $|a|$ and let $g = U^{-1} \circ h \circ U$. Then $g \in \text{Aut}(B^n)$, $g(0) = a$, and $g(-a) = 0$. The transformation $g$ can be expressed as

\begin{equation}
 g(z) = \frac{P_a(z) + a}{1 + z \cdot \bar{a}} + \frac{\sqrt{1 - |a|^2}}{1 + z \cdot \bar{a}} Q_a(z),
\end{equation}

where

\[ P_a(z) = \frac{z \cdot a}{|a|^2} a, \quad Q_a(z) = z - P_a(z). \]

It follows that every point in $B^n$ is mapped to the origin by some member of $\text{Aut}(B^n)$, hence the $\text{Aut}(B^n)$ action is transitive and $B^n$ is a homogeneous domain. Thus $\dim \text{Aut}(B^n) = 2n$.

Since each element of the unitary group $U(n)$ is an automorphism of $B^n$ that fixes the origin, we see that $U(n)$ is naturally a subgroup of $\text{Aut}(B^n)$. Thus $\dim \text{Aut}(B^n) = n^2$. Therefore $\dim \text{Aut}(B^n) = n^2 + 2n$.

The group $U(n, 1)$ is the group of linear transformations under which the form $X_1 \bar{X}_1 + \cdots + X_n \bar{X}_n - X_0 \bar{X}_0$ is invariant. So

\[ U(n, 1) = \{ W \in GL(n + 1, \mathbb{C}) : W^T \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} W = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \}. \]

In homogeneous coordinates, $B^n$ is given by

\[ X_1 \bar{X}_1 + \cdots + X_n \bar{X}_n - X_0 \bar{X}_0 < 0. \]

Each $W = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$ in $U(n, 1)$ gives a map $g_W \in \text{Aut}(B^n)$, where

\begin{equation}
 g_W(z) = \frac{Az + b}{cz + d}.
\end{equation}

The map $g_W$ is the identity map if and only if $W \in S^1 := \{ e^{it} I : t \in \mathbb{R} \}$. Thus $PU(n, 1) := U(n, 1)/S^1$ is considered a subgroup of $\text{Aut}(B^n)$. Since $(n + 1)^2 - 1 = \dim PU(n, 1) \leq \dim \text{Aut}(B^n) = 2n + n^2$, we see that $\text{Aut}(B^n) = PU(n, 1)$.

If $b = (t, 0, \ldots, 0)$, $t > 0$, $c = b^T$, then $A = \begin{pmatrix} \alpha \sqrt{1 + t^2} & 0 \\ 0 & V \end{pmatrix}$, $d = \alpha \sqrt{1 + t^2}$, where $V \in U(n - 1)$, $|\alpha| = 1$. Take $\alpha = 1$, $V = I_{n-1}$ to obtain $A = \begin{pmatrix} \sqrt{1 + t^2} & 0 \\ 0 & I_{n-1} \end{pmatrix}$ and
\[ d = \sqrt{1 + t^2}. \] Then \( g_{W} \) is given by (1) with \( \lambda = t/\sqrt{1 + t^2} \). The map in (3) is the transformation given by (2) with \( a = d^{-1}b \).

3. The automorphism group of the entire space \( \mathbb{C}^n, n \geq 2 \) is not finite dimensional. Say, for \( \mathbb{C}^2 \) it contains shears \((w_1 = z_1; w_2 = z_2 + f(z_1))\) for any entire function \( f \).

4. More examples one can find in A. Isaev’s book [36].

5. One more interesting result. It can be proved that if \( D \) is a bounded domain in \( \mathbb{C}^n \) and \( \dim(\text{Aut}(D)) = n^2 + 2n \), then \( D \) is biholomorphic to \( B^n \). A. Isaev gave explicit classification of all domains \( D \) in \( \mathbb{C}^n \) for which \( n^2 - 1 \leq \dim(\text{Aut}(D)) < n^2 + 2n \). These results are fully presented in [36].

5.3. Greene-Krantz conjecture. In case of strictly pseudoconvex domains with smooth boundary only the unit ball has a NON-compact group of holomorphic automorphisms. This was proved by B. Wong [59]. The non-compactness of \( \text{Aut}(D) \) means that for at least one point \( z_0 \in D \) the orbit \( \text{Aut}(D)(z_0) \) has an accumulation point \( p \) on the boundary. J. P. Rosay [47] extended Wong’s theorem by proving that if \( p \) is a strictly pseudoconvex point of \( \partial D \) then \( D \) is biholomorphic to the ball. The proof of Wong-Rosey theorem can be done by the scaling method ([40], p. 227). In the same survey one can find various versions of this theorem (even for infinite dimensional cases).

Because of these theorems the description of smoothly bounded domains with non-compact automorphism group becomes interesting. All the known examples of such domains reveal that the accumulation point on \( \partial D \) is of “finite type in the sense of Kohn, D’Angelo, and Catlin”. (A boundary point of a domain in \( \mathbb{C}^2 \) is of finite type if the boundary has finite order of contact with complex manifolds through the point; precise definition for \( \mathbb{C}^n \) can be found in [40]). In 1991 R. Greene and S.Krantz made the following conjecture [30]:

**Conjecture.** Let \( D \) be a smoothly bounded domain in \( \mathbb{C}^n \). Suppose that \( x \in D \) has a boundary orbit accumulation point for the automorphism group in the sense that there are automorphisms \( \phi_j \in \text{Aut}(D) \) and a point \( p \in \partial D \) such that \( \phi_j(x) \to p \) as \( j \to \infty \). Then \( p \) is a point of finite type.

There have been many attempts to resolve this conjecture, only partial results have been obtained by now; for a more detailed discussion on it see [40].

5.4. Narasimhan’s question. In this section we will again use the notation \( \text{Aut}_D \) instead of \( \text{Aut}(D) \).

A bounded domain \( \Omega \subset \mathbb{C}^n \) is said to have Property \( N \) if there exists a compact subset \( K \) of \( \Omega \) with the property that for each \( z \in \Omega \) there is an \( f \in \text{Aut}_\Omega \) such that \( f(z) \in K \). For \( z \in \Omega \) and a subgroup \( \Gamma \) of \( \text{Aut}_\Omega \), \( \Gamma(z) := \{f(z) : f \in \Gamma\} \) is the \( \Gamma \)-orbit of \( z \). If \( S \subset \Omega \), then \( \Gamma(S) \) denotes the union of the \( \Gamma \)-orbits of the points in \( S \):

\[
\Gamma(S) := \bigcup_{\varepsilon \in S} \Gamma(\varepsilon) = \{f(z) : z \in S, f \in \Gamma\}.
\]

Thus Property \( N \) is equivalently defined as follows: \( \Omega \) is said to have property \( N \) if there is a \( K \subset \subset \Omega \) such that \( \text{Aut}_\Omega(K) = \Omega \). The following is a question by R. Narasimhan:

**Question 1.** If \( \Omega \) is a bounded domain in \( \mathbb{C}^n \) with Property \( N \), is \( \Omega \) necessarily a homogeneous domain?
A *discrete* subgroup of Aut_Ω is a subgroup of Aut_Ω that is discrete in the compact open topology, the default topology of Aut_Ω. Recall that we say a subgroup Γ of Aut_Ω acts on Ω freely if each f ∈ Γ is fixed-point free.

The following theorem is by S. Frankel [16].

**Theorem 5.2.** Let Ω be a convex hyperbolic domain in C^n and suppose that there is a discrete subgroup Γ ⊂ Aut_Ω such that Ω/Γ is compact. Then Ω is biholomorphic to a bounded symmetric domain.

Frankel first proved a distortion theorem for convex holomorphic embeddings and used this to reduce a complex analysis problem to one in affine geometry. Then he applied rescaling to produce a continuous family of automorphisms. His particular technique of boundary localization was very different from what had gone before, and he called it the rescale blow-up.

Kazhdan conjectured that each irreducible bounded domain which admits both a compact quotient and a one-parameter group of holomorphic automorphisms must be biholomorphic to a bounded symmetric domain. Frankel [16] first confirmed the conjecture for bounded convex domains. Subsequent work by Nadel [44] and Frankel [17] proved it in general.

**Theorem 5.3.** Let M be a compact complex manifold. Then the group Aut_M is a complex Lie transformation group and its Lie algebra consists of holomorphic vector fields on M.

The above theorem is due to Bochner and Montgomery [5, 6]. For bounded domains in C^n we have the following theorem of H. Cartan [9, 11].

**Theorem 5.4.** Let D be a bounded domain in C^n. Then the group Aut_D is a Lie transformation group and the isotropy subgroup Aut_Dz at any point z ∈ D is compact. If X is in the Lie algebra of Aut_D, then JX is not in the Lie algebra of Aut_D.

**Theorem 5.5.** (Kobayashi [39, p. 81] Let M be a hyperbolic manifold. Then the group Aut_M is a Lie transformation group and the isotropy subgroup Aut_Mx of M at any point x ∈ M is compact.

The essential reason for the above three theorems is the following

**Theorem 5.6.** (Bochner and Montgomery [6]) Let G be a locally compact group of differentiable transformations of a manifold M. Then G is a Lie transformation group.

Theorem 5.5 is also based on the following early result of van Danzig and van der Waerden [13].

**Theorem 5.7.** Let M be a connected, locally compact metric space and G_M the group of isometries of M. Then G_M is locally compact with respect to the compact-open topology.

### 6. Determining sets and fixed points

Let M be a complex manifold, f : M → M a holomorphic map. z_0 ∈ M is a fixed point for f if f(z_0) = z_0.

The following is a result in the classical function theory [43, 45]: if f : M → M is a conformal self-mapping of a plane domain M which fixes three distinct points then f(ζ) = ζ.
This one-dimensional result is true even for endomorphisms of a bounded domain $D \subset \subset \mathbb{C}$. To prove this one needs to first use the well known theorem, stating that if an endomorphism of $D$ fixes two distinct points, then it is an automorphism; and then use the above cited [43, 45] theorem.

We now introduce two notions to discuss how this result can be extended for higher dimensions.

For a complex manifold $M$ let $H(M, M)$ be the set of holomorphic maps from $M$ to $M$, i.e., the set of endomorphisms of $M$. The group of holomorphic automorphisms of $M$, $\text{Aut}(M)$ is a subset of $H(M, M)$.

**Definition 6.1.** A set $K \subset M$ is called a **determining subset** of $M$ with respect to $\text{Aut}(M)$ ($H(M, M)$ resp.) if, whenever $g$ is an automorphism (endomorphism resp.) such that $g(k) = k \forall k \in K$, then $g$ is the identity map of $M$.

So, any three points of a plane domain $D$ form a determining set for $\text{Aut}(D)$ as well as for $H(D)$.

The other notion is $\text{Fix}(f)$, it denotes the set of fixed points $\{x \in M \mid f(x) = x\}$ of $f$.

So, if $M$ is a plane domain, and $\text{Fix}(f)$ is discrete, then the cardinality of this set $\#\text{Fix}(f) \leq 2$ for any $f$ according to our remarks at the beginning of this section.

We now have two classes of problems to investigate. First: the description and properties of determining sets for various complex manifolds. There are many results in classical analysis and topology proving the non-emptiness of $\text{Fix}(f)$, or finding it for a given function. Our second class of problems to discuss is different from those: for a given $M$ which sets can be $\text{Fix}(f)$ for some holomorphic $f \in H(M, M)$.

6.1. **Determining sets.** The notion of determining sets was first introduced in [25]. That paper was an attempt to find a higher dimensional analog of the above one-dimensional result. Determining sets (for automorphisms and endomorphisms) in case of bounded domains in $\mathbb{C}^n$ have been further investigated in the following papers [26]-[29],[38, 54, 55].

Let’s first look now at some examples of discrete non-determining and determining sets (from [25]).

**Example 6.2.** Let $A = \{z \in \mathbb{C} : 1/2 < |z| < 2\}$. This is an annulus in the plane. The map $\tau(z) = 1/z$ has two fixed points (i.e., 1 and $-1$), yet $\tau$ is not the identity mapping, so these points are not a determining set. Since the $\text{Aut}(A)$ is well known one can check that any two points in general position in $A$ do form a determining set.

**Example 6.3.** In $\mathbb{C}^2$ consider a shear of the form $\tau(z, w) = (z, w + \phi(z))$, where $\phi$ is any entire function on the plane. Then $\tau$ is a biholomorphic map of $\mathbb{C}^2$. If $\phi$ has infinitely many distinct zeros then $\tau$ will have infinitely many fixed points, even though $\tau$ is not the identity. So, the set of these zeros does not form a determining set for $\text{Aut}(\mathbb{C}^n)$.

By contrast, any biholomorphic (conformal) map of $\mathbb{C}$ that fixes two points must be the identity. So, any two distinct points in $\mathbb{C}$ form a determining set for the $\text{Aut}(\mathbb{C})$.

**Example 6.4.** It can be shown that a biholomorphic map of the unit ball $B^n$ in $\mathbb{C}^n$ that fixes $n + 1$ points in general position (in the usual sense of topology) must be the identity. One may check this by using the description of the automorphism group of the ball given in the previous section. We leave the details to the interested reader. So, this set of $n + 1$ points forms a determining set for the ball.
We also note that no set of \( n \) points in \( B^n \) forms a determining set. Indeed, let \( p_1, ..., p_n \) be the \( n \) points. Since the ball is a homogeneous domain, we may consider \( g \in \text{Aut}(B^n) \) such that \( g(p_1) = 0 \). Now the set \((g(p_1), g(p_2), ..., g(p_n))\) lies in a linear space \( L \) of dimension \( \text{dim}(L) \leq n - 1 \). Therefore there is a rotation \( f \in \text{Aut}(B^n) \) that keeps all the points of \( L \) fixed. So, the automorphism \( h = g^{-1}fg \in \text{Aut}(B^n) \) is not an identity and fixes all the \( n \) points \((p_1, ..., p_n)\).

**Example 6.5.** Consider the domain \( U_m = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^{2m} < 1 \} \), any integer \( m \geq 2 \). Then any automorphism of \( U_m \) that fixes two points in general position must be the identity. This result follows because the automorphism group of \( U_m \) is well-known to consist only of rotations in each variable separately.

Contrast this example with the result from the last example (for the unit ball in \( \mathbb{C}^n \)).

**Example 6.6.** Let \( U_m \) be one of the domains from the last example. Let \( V \) be any rigid domain in \( \mathbb{C}^n \) (here rigid means that the domain has no automorphisms except the identity). Then, for an arbitrary chosen pair of points \( z \in U_m, w \in U_m \), and an arbitrary \( x \in V \), any automorphism of \( U_m \times V \) which fixes both \((z, x)\) and \((w, x)\) will be the identity. For instance, the points \( z = ((1/2, 0), x) \) and \( w = ((0, 1/2), x) \) will do.

We will present now a few results when the determining set \( K \) is *discrete* and refer the interested reader to the above mentioned papers for more results and unsolved problems.

We will need two more notions. Let \( M \) be a complex manifold. Let \( W_s(M) \) denote the set of \( s \)-tuples \((x_1, \ldots, x_s)\), where \( x_j \in M \), such that \( \{x_1, \ldots, x_s\} \) is a determining set with respect to \( \text{Aut}(M) \). Similarly, \( \widehat{W}_s(M) \) denotes the set of \( s \)-tuples \((x_1, \ldots, x_s)\) such that \( \{x_1, \ldots, x_s\} \) is a determining set with respect to \( H(M, M) \). So \( \widehat{W}_s(M) \subseteq W_s(M) \subseteq M^s \). We now introduce two values \( s_0(M) \) and \( \widehat{s}_0(M) \). In case \( \text{Aut}(M) = \text{id} \), \( s_0(M) = 0 \), otherwise \( s_0(M) \) is the least integer \( s \) such that \( W_s(M) \neq \emptyset \). If \( W_s(M) = \emptyset \) for all \( s \) then \( s_0(M) = \infty \). Analogously symbol \( \widehat{s}_0(M) \) denotes the least integer \( s \) such that \( \widehat{W}_s(M) \neq \emptyset \), if no such integer exists (i.e. \( \widehat{W}_s(M) = \emptyset \) for all \( s \)) then \( \widehat{s}_0(M) = \infty \). In all cases \( s_0(M) \leq \widehat{s}_0(M) \).

1. For hyperbolic manifolds of dimension \( n \) the following estimate holds \( \widehat{s}_0(M) \leq n + 1 \).

First we note that this estimate for bounded domains in \( \mathbb{C}^n \) was proved by J.P. Vigne (see [55]). In that paper a more precise theorem is proved. Let \( a \) be a point in a bounded domain \( D \). Then there is an open set \( U \subset D^n \) such that \((a, \ldots, a) \in U \) and for all \((z_1, \ldots, z_n) \in U, (z_1, \ldots, z_n) \in \widehat{W}_s(M) \).

For a general hyperbolic manifold \( M \) one may consider a small Kabayashi ball \( b \in M \) such that it is biholomorphic to a domain \( D \in \mathbb{C}^n \). Pick a point \( a \in M \). Let \( f : M \to M \) be a holomorphic map such that \( f(a) = a \). Consider a small Kobayashi ball \( b = b(a, \epsilon) \) whose closure is compact in \( M \), and such that \( b \) is biholomorphic to a bounded domain \( D \subset \mathbb{C}^n \); let \( h : b \to D \) be such a biholomorphic map. Note that since the Kobayashi distance is non-increasing under holomorphic maps, we have \( f : b \to b \), and therefore \( g = hfh^{-1} : D \to D \). By using the above mentioned result, one can pick \( n \) points \( z_1, \ldots, z_n \in D \), such that \( Z = (h(a), z_1, \ldots, z_n) \in \widehat{W}_{n+1}(D) \). Consider the set of \( n + 1 \) points \( h^{-1}(Z) = (a, h^{-1}(z_1), \ldots, h^{-1}(z_n)) \subset b \). If our function \( f \in H(M, M) \) (in addition to \( a \)) is also fixing all points \( h^{-1}(z_j) \), i.e. \( f|_{h^{-1}(Z)} = id \), then \( g|_Z = id \) and therefore \( g = id \). We conclude that \( f|_b = id \), and consequently \( f = id \). So, \( h^{-1}(Z) \in \widehat{W}_{n+1}(M) \), and therefore \( \widehat{s}_0(M) \leq n + 1 \).
2. The above statement implies same inequality for automorphisms of a hyperbolic manifold $M$, $s_0(M) \leq n + 1$.

However for automorphisms much more information can be provided. $s_0(M)$ depends on how large the group $\text{Aut}(M)$ is. If $M$ is a bounded domain in $\mathbb{C}^n$ ($\text{dim}(M) = n$) then the general estimate ($s_0(M) \leq n + 1$) can be refined to $s_0(M) \leq n$ for domains that are not biholomorphic to the unit ball $B^n \subset \mathbb{C}^n$, and the only hyperbolic manifolds for which $s_0(M) = n + 1$ are those biholomorphic to the ball. This gives a characterization of the ball in $\mathbb{C}^n$.

3. If a positive integer $s \geq s_0(M)$, then $W_s(M) \neq \emptyset$, so there are $s$ points such that if an automorphism of $M$ fixes these points it will fix any point of $M$. Now the question arises whether the choice of these $s$ points is generic. The answer is positive for any hyperbolic manifold $M$: $W_s(M) \subseteq M^s$ is open and dense if not empty. Similar topological properties for the determining sets of endomorphisms of a general hyperbolic manifold do not hold.

6.2. Isolated fixed point sets, cardinality. In classical mechanics the following Euler’s theorem is well known: the general displacement of a rigid body with one point fixed is a rotation about some axis. So, if one considers an orientation-preserving isometry of a domain in $\mathbb{R}^3$ fixing one point, the fixed point set of this isometry will necessarily contain at least a segment, so the fixed point set cannot be a discrete set. In the euclidean space $\mathbb{R}^n$, one can always find a domain which has a euclidean isometry with exactly one fixed point, however for any $n$, if an isometry of a domain in $\mathbb{R}^n$ has two fixed points it will force the existence of at least a segment to belong to the fixed point set, and so this set will be at least one dimensional.

Switching to complex analysis, we remark that any holomorphic automorphism of a bounded domain in $\mathbb{C}^n$ (or in general, hyperbolic manifold) is an isometry in an invariant metric, so an Euler type statement is certainly meaningful, that is if this automorphism has a discrete fixed point set one can inquire what its cardinality and structure might be. To describe this more precisely, let $f : M \rightarrow M$ be a holomorphic self-map of a complex manifold $M$. Suppose that $\text{Fix}(f)$ is discrete. We shall examine mostly two questions. First, how large this set can be for specific cases: $M$ is a bounded domain in $\mathbb{C}^n$, a hyperbolic manifold, etc., while $f$ is a holomorphic automorphism or endomorphism. Second, the structure of $\text{Fix}(f)$, namely which points of $M$ could form such a set for some holomorphic self-map of $M$. Everywhere below we consider only holomorphic self-maps (automorphisms or endomorphisms) of various complex manifolds, and for the sake of compactness the word holomorphic may be omitted.

In examining the cardinality of a discrete fixed point set, let’s first consider the situation in one dimension. For a bounded domain $D \subset \mathbb{C}$ the discrete fixed point set of a holomorphic map $f : D \rightarrow D$ can have no more than two points. This follows from the above mentioned observation: any set of three points in $D$ must be a determining set for endomorphisms. An annulus gives an example of a domain that has an automorphism with exactly two fixed points.

In $\mathbb{C}^n$ the situation is not yet completely clear. Here are a few statements we know.

1. For a convex domain one has the following theorem: the isolated fixed point set of any endomorphism consists of at most one point. This statement follows from the main theorem in [56]: such a set has to be connected. The proof is based on establishing that in Bergman
metric there is a unique geodesic connecting two fixed points, and it (the geodesic) will
then also belong to the fixed point set.

2. For a bounded strictly pseudoconvex domain $D$ in $\mathbb{C}^n$ with real analytic boundary the
number of points in a discrete fixed point set of an automorphism is finite. Moreover, there
is a number $m = m(D)$ such that $\#(\text{Fix}(f)) \leq m$.

Here’s a proof of this statement. If $D$ is biholomorphic to the ball or if $n = 1$, then
the statement is clear. Assume that $n \geq 2$ and $D$ is not biholomorphic to the ball. By a
theorem in [57], there is a neighborhood $U$ of $g$ with smooth boundaries so that
$D \subset U_3 \subset U_2 \subset U_1$. For every $h \in \text{Aut}(D)$ in some neighborhood of $g$, $h(\partial U_2)$ is so close to $G(\partial U_2)$ that $h(\partial U_2) \cap g(U_3) = \emptyset$. Since $h(U_2)$ is a connected component of $\mathbb{C}^n \setminus h(\partial U_2)$ and since $h(U_2) \supset D$, we see that $h(U_2) \supset g(U_3)$ for every $h \in \text{Aut}(D)$ in some neighborhood of $g$. Since $\text{Aut}(D)$ is compact, there is a neighborhood $Q$ of $D$ such that $Q \subset g(U_1)$ for each $g \in \text{Aut}(D)$. Let $U$ be the interior of the intersection of the sets $g(U_1), g \in \text{Aut}(D)$. Then $U \supset Q$ and $g(U) = U$ for each $g \in \text{Aut}(D)$. I.e., each automorphism of $D$ is also an automorphism of $U$. There is a
finite cover of open sets $\{V_j : j = 1, \ldots, m\}$ of $D$ such that each pair of points in a $V_j$ is
connected by a unique distance-minimizing geodesic with respect to the Bergman metric
of $U$. Let $f \in \text{Aut}(D)$. If $f$ fixes two points in a $V_j$, $f$ must fix each point on the unique
distance-minimizing geodesic connecting the two points. Consequently, each $V_j$ contains
at most one isolated fixed point of $f$. Therefore, the number of isolated fixed points of $f$ is
$\leq m$.

3. Must the cardinality of an isolated fixed point set of an automorphism or endomor-
phism be bounded by a number depending only on the dimension of the manifold under
consideration? For endomorphisms of bounded domains in $\mathbb{C}^n$ the answer is negative. It
is also negative for automorphisms of a general hyperbolic manifold and the entire $\mathbb{C}^n$.
However, for an automorphism of a bounded domain in $\mathbb{C}^n$ the answer is not yet clear.
Let’s consider several examples demonstrating some of the results.

Example 6.7. For any $k \in \mathbb{N}$, there exists a bounded domain $D \subset \mathbb{C}^n$, $n \geq 2$, and a
holomorphic endomorphism $f : D \to D$, such that $\#(\text{Fix}(f)) = k$.

Proof. Without any loss of generality we can present an example for $n = 2$. Let $S$ be the
open Riemann surface in $\mathbb{C}^2 : S = \{(x, y) \in \mathbb{C}^2|y^2 = (x - a_1)\ldots(x - a_k)\}$, where $a_1, \ldots, a_k$ are $k$ distinct points in $\mathbb{C}$. The restriction $g$ of $(x, y) \to (x, -y)$ to $S$ has exactly $k$ fixed points. Following [[34], VIII, C8, p.257] there exists a holomorphic retraction $\rho : V \to S$ of
an open neighborhood $V$ of $S$ onto $S$. Now the mapping $f := g \circ \rho : V \to V$ has
exactly $k$ fixed points. Of course $V$ is not bounded, but we can consider a bounded open
set $W \subset V, (a_s, 0) \in W$ for all $s = 1, \ldots, k$ and such that $g(W) = W$. This bounded domain
will have the same property.

Example 6.8. There exists a hyperbolic manifold with a holomorphic automorphism whose
fixed point set is discrete and consists of an infinite number of points.

Proof. Consider the submanifold $X$ of $D^2$ defined by $y^2 = B(x)$, where $D$ is the open
unit disc and $B$ is a Blaschke product with an infinite number of zeroes, the restriction to
$X$ of the map $(x, y) \to (x, -y)$ is an automorphism of $X$ and has an infinite number of
isolated fixed points.
Example 6.9. For any \( n \geq 2 \) and any \( k \in \mathbb{N} \), there exists a polynomial automorphism \( f \) of \( \mathbb{C}^n \), such that \( \#(\text{Fix}(f)) = k \). Moreover, let \( n \geq 2; p_1, p_2, \ldots, p_k \) are \( k \) distinct points in \( \mathbb{C}^n \). Then there exists a polynomial automorphism \( g \in \text{Aut}(\mathbb{C}^n) \) such that \( \text{Fix}(g) = \{p_1, p_2, \ldots, p_k\} \).

Proof. Let \( a_1, \ldots, a_k \) be \( k \) distinct complex numbers. Consider the map \( H : \mathbb{C}^n \to \mathbb{C}^n \) given by

\[
\begin{align*}
w_1 &= z_1 + z_2 + (z_1 - a_1)(z_1 - a_2) \ldots (z_1 - a_k) \\
w_2 &= z_2 + (z_1 - a_1)(z_2 - a_2) \ldots (z_2 - a_k) \\
w_s &= iz_s \quad \text{for all } s = 3, \ldots, n
\end{align*}
\]

This map is an automorphism, whose fixed point set is the set of the following \( k \) points:

\[(a_1, 0, \ldots, 0), (a_2, 0, \ldots, 0), \ldots, (a_k, 0, \ldots, 0)\].

Now \( p_j = (a_j, b_j), a_j \in \mathbb{C}, b_j \in \mathbb{C}^{n-1} \). Without any loss of generality we assume that the \( a'_j \)'s are all distinct (in case they are not, one can first use an invertible linear transformation of \( \mathbb{C}^n \) to achieve this). Consider the polynomial transformation \( F : w_1 = z_1, w' = z' + f(z_1) \), where \( f : \mathbb{C} \to \mathbb{C}^{n-1} \) is the Lagrange interpolation polynomial map satisfying \( f(a_j) = b_j, w' = (w_2, \ldots, w_n) \). Then \( F(a_j, 0) = p_j, j = 1, \ldots, k \), and \( F \in \text{Aut}(\mathbb{C}^n) \). Now the automorphism \( g = F \circ H \circ F^{-1} \) is such that \( \text{Fix}(g) = \{p_1, p_2, \ldots, p_k\} \).

Some problems.

1. Let \( D \) be a bounded domain in \( \mathbb{C}^n, f \in \text{Aut}(D) \), and \( \text{Fix}(f) \) is a discrete set. Can \( \#(\text{Fix}(f)) = \infty? \)

If one considers the domain \( D \subset \mathbb{C}^n \) which is a direct product of \( n \) annuli, one can then find an \( f \in \text{Aut}(D) \) with \( \#(\text{Fix}(f)) = 2n \). So, the next natural question is

2. Let \( n \geq 2, D \) be a bounded domain in \( \mathbb{C}^n \), with a piecewise smooth boundary, \( f \in \text{Aut}(D) \), and \( \text{Fix}(f) \) is a set of isolated points. Can \( \#(\text{Fix}(f)) \geq 2n + 1? \) (As noted earlier, for \( n = 1 \) the answer is negative). A more restricted version of this question is

3. Is there a number \( m \) such that for any strongly pseudoconvex domain \( D \subset \subset \mathbb{C}^n, \partial D \in C^\infty, \) and \( f \in \text{Aut}(D) \), if \( \text{Fix}(f) \) is a set of isolated points, then \( \#(\text{Fix}(f)) \leq m \), where \( m = m(n) \) (i.e. \( m \) depends on the dimension only)?

6.3. Fixed point sets consisting of one or two points. In this section we'll discuss which subsets of a manifold \( D \) can be \( \text{Fix}(f) \) for some holomorphic automorphism or endomorphism \( f : D \to D \). As the title suggests we'll consider two cases.

First we consider the case when every single point of a domain is the \( \text{Fix}(f) \) for a suitable holomorphic \( f \).

Theorem 6.10. \([28, \text{Theorem 2.1}] \) If every point of a hyperbolic manifold \( D \) is a fixed point set for some holomorphic automorphism of \( D \), then \( D \) is a homogeneous manifold.

Proof. 1. First we note that the theorem will follow from a local statement: let \( x \in D \), then there exists a neighborhood \( U_x \) of \( x \) such that for any \( y \in U_x \) there is a \( g \in \text{Aut}(D) \) such that \( g(y) = x \). Indeed, if this is true consider two arbitrary points \( a, b \in D \), connect them by a compact path \( L \), cover \( L \) by a finite number of \( U_x, x \in L \), and one can obtain an \( f \in \text{Aut}(D) \), such that \( f(a) = b \).

2. We now prove the local statement. Let \( x \in D \). By \([29]\), for each point \( x \in D \) there is an invariant Hermition metric in some neighborhood of the orbit \( G(x) \), where \( G = \text{Aut}(D) \). Consider a small enough ball \( b(x, \epsilon) \) in that metric with center \( x \) and radius \( \epsilon, \epsilon > 0 \) will be determined by the construction later. Let \( y \in b(x, \epsilon) \); consider
the orbit $O(y) = \{ z \in D : \exists g \in Aut(D), g(y) = z \}$. Consider now a point $p \in O(y)$, such that $d(x,p) = d(x, O(y))$, where $d(.,.)$ denotes the distance function induced by the local invariant metric. Clearly, $p \in b(x, \epsilon)$. If $p = x$, there is nothing to prove; otherwise consider a small ball $b_1$ of radius $< d(x,p)$ that lies inside $b(x,d(x,p))$, and such that $\partial b_1 \cap \partial b(x, d(x,p)) = p$. This construction is possible if $\epsilon$ is small enough, fixing such an $\epsilon = \epsilon(x)$, we denote $b(x, \epsilon) = U_x$.

We observe that $O(y) \cap b(x, d(x,p)) = \emptyset$. Let $q$ denote the center of the ball $b_1$. By the assumption of the theorem there exists an $h \in Aut(D)$ whose fixed point set is $q$. Now $h(p) \neq p$, and $h(p) \in \partial b_1$, since $h(\partial b_1) = \partial b_1$. We now conclude that $h(p) \in O(y) \cap b(x, d(x,p))$, which contradicts the previous observation that this intersection is empty. Therefore $x = p \in O(y)$, and the theorem has been proved. □

We now provide the following example.

**Theorem 6.11.** There exists a domain $D$ in $\mathbb{C}$ with infinite number of points each of which is the fixed point set for a holomorphic automorphism of $D$.

**Proof.** Consider $D = \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \Delta(n,1/3)$ where $\Delta(n,1/3)$ is a disk with center at $n \in \mathbb{Z}$ and radius $1/3$. Consider $f_k : z \mapsto (-z + (2k + 1))$. Then for any $k \in \mathbb{Z}, f_k \in Aut(D)$, and its fixed point set consists of one point $Fix(f) = \{k + 1/2\}$. □

Let’s now consider pairs of points as fixed point sets. Though such domains exist, no domain can have too many pairs of distinct points as a fixed point set for an automorphism.

**Theorem 6.12.** Let $D \subset \subset \mathbb{C}^n$. The set $N \subset D^2$ of all pairs, each of which cannot be a fixed point set for a holomorphic automorphism of $D$, contains a full measure set in $D^2$.

It follows from the following two Lemmas. The first is a classical statement (see [8] p. 80; also [54] thm 2.3) proving that for $z \in D$ and its isotropy subgroup $I_z = \{ f(z) = z, f \in Aut(D) \}$ there is a local system of coordinates where each $f \in I_z$ is a linear map.

**Lemma 6.13.** There exists a holomorphic map $\phi : D \to \mathbb{C}^n$ such that $\phi(z) = 0, \phi'(z) = id$, and for all $f \in I_z$ one has $\phi \circ f = f'(z) \circ \phi$.

The theorem will now follow from the second lemma.

**Lemma 6.14.** Let $D \subset \subset \mathbb{C}^n, a \in D$. Then there exists a complex analytic set $Z \subset D, (\dim Z < n)$, such that if $b \in D \setminus Z$ then the two points $\{a, b\}$ are such that for any automorphism $f$ fixing these two points, the fixed point set of $f$ is at least one (complex) dimensional.

**Proof.** Using the previous lemma we first find the function $\phi$ for the point $a$. Let $Z = \{ z \in D | \phi(z) = 0 \}$. If $b \in D \setminus Z$, then suppose $f \in Aut(D)$ and $f$ fixes both points $a$ and $b$. We have $f'(a) \cdot \phi(b) = \phi(f(b)) = \phi(b)$. Since by choice $\phi(b) \neq 0$, and $\phi$ is biholomorphic in the neighborhood $U$ of $a$, for a number $\lambda, |\lambda| > 0$, small enough, there exists a point $c \in U \subset D, c \neq a, \phi(c) = \lambda \phi(b)$, and $f(c) \in U$. Now $\phi(f(c)) = f'(a) \cdot \phi(c) = f'(a) \cdot \lambda \phi(b) = \lambda \phi(b) = \phi(c)$.

Since $\phi$ is biholomorphic in $U$ we have $f(c) = c$. □
7. Approximate biholomorphisms

7.1. “Approximate” biholomorphism: exhaustion. As noted in the introduction two randomly picked domains in \( \mathbb{C}^n \) are likely to be non-equivalent (=non-biholomorphic). Can they be “approximately” equivalent? Let’s make this question precise. Consider a sequence of bounded domains \( \{D_k\}_{k=1}^{\infty} \subset \mathbb{C}^n \), such that all \( D_k \subset D \) and \( \lim_{k \to \infty} D_k = D \) in some topology of domains in \( \mathbb{C}^n \). So, \( D \) can be approximated by \( D_k \) for large \( k \). Suppose now that there is a domain \( G \subset \subset \mathbb{C}^n \) such that each \( D_k \) is a biholomorphic image of \( G \), \( f_k : G \to D_k \) a biholomorphic map onto \( D_k \). In this case we will say that \( D \) can be exhausted by \( G \). So, \( G \) is approximately equivalent to \( D \).

So, if given a bounded domain \( G \) in \( \mathbb{C}^n \) what can it exhaust? The list \( \Lambda(G) \) of these domains is of course a biholomorphic invariant. The range varies widely depending on \( G \), and we’ll mention several examples. By definition \( \Lambda(G) \) contains \( G \). We will show that if \( G \) is homogeneous then \( \Lambda(G) \) consists only of \( G \) itself:

**Theorem 7.1.** If \( G, D \) are bounded domains in \( \mathbb{C}^n \), \( G \) is homogeneous and \( D \) can be exhausted by \( G \) (in the Hausdorff topology), then \( D \) is biholomorphic to \( G \).

By the way, since the ball and the polydisc in \( \mathbb{C}^n \) for \( n > 1 \) are non-biholomorphic it follows that neither can exhaust the other and there should be “largest” imbedding of each into the other. The precise estimates have been obtained by H. Alexander [2], which can be considered as another proof of the Poincare’s Example.

If \( G \) is a smooth, bounded strictly pseudoconvex domain, then \( \Lambda(G) \) has only the following elements: \( G \) and the unit ball \( B^n \) [21]. Comparing this example with the above theorem shows that the notion of exhaustion is not symmetric: it happens that \( D \in \Lambda(G) \) but \( \Lambda(D) \) does not contain \( G \).

For half-the ball in \( \mathbb{C}^n \), \( n \geq 2 \), \( S = \{z : z = (z_1, \ldots, z_n) \in B^n, Re(z_n) > 0\} \), \( \Lambda(S) \) contains the unit ball and the polydisk [19]. This, by the way shows that \( S \) is not biholomorphically equivalent to any smooth, bounded strictly pseudoconvex domain (though, of course, this can be proved by other means).

The above theorem shows that a homogeneous domain exhausts only itself. On the other side of the spectrum is the following domain.

**Theorem 7.2.** (Fridman [18]). There exists a universal domain \( U \subset \subset \mathbb{C}^n \) which can exhaust any other bounded domain in \( \mathbb{C}^n \) (in the Hausdorff topology).

So, this domain is “almost” equivalent to any other domain in \( \mathbb{C}^n \). Any domain/set that has the approximation property described in this theorem we will call a universal domain/set.

As we noticed at the start of this exposition, there is no version of the Riemann mapping theorem in \( \mathbb{C}^n \). The above statement can be considered the approximate Riemann mapping theorem for any \( \mathbb{C}^n \).

A short explicit construction one can find in [18], it will not be repeated here. We’ll make a few remarks concerning the universal domain.

There is a great flexibility of constructing this domain. I.e. such a domain can have the automorphism group isomorphic to \( \mathbb{Z}_k \), for any \( k \in \mathbb{N} \). Also one can construct many such non-biholomorphic domains; this gives an example of two domains that can exhaust each other but are non-equivalent.
There is a great variety of universal domains. And most of them have one curious property we are about to describe. The exposition is short and elementary, but the statements are useful when dealing with general biholomorphic mappings and therefore we are including them here. It would be helpful for the interested reader to know the construction before reading about the following property of $U$. It is as follows. There is a designated point $p \in \partial U$ such that to approximate a domain $G \subset \mathbb{C}^n$ with a given precision $\epsilon$, one has to find a small $\delta$ and a biholomorphic imbedding $T : U \to G$, such that the $\delta$-neighborhood $W_\delta = B(p, \delta) \cap U$ will “blow up” to cover most of $G$ while $T$ will squeeze everything else outside that neighborhood in $U$ almost to a point. This will accomplish the goal: $T(W_\delta) \subset G$ will be $\epsilon$-close to $G$, while the rest of $T(U \setminus W_\delta)$ will be small enough not to create a larger approximation mistake. We use the notation $B(p, \delta)$ for a ball at center $p$ and radius $\delta$.

So, by an obvious association we can call $p$ a “source” of all domains in $\mathbb{C}^n$, and the described property a “Big Bang property”. One can also express this property by stating that for any $\delta > 0$ the set $B(p, \delta) \cap U$ is also a universal domain. Is the existence of such a “source” necessary for a universal domain? We prove the following.

**Theorem 7.3.** Let $U$ be a universal domain. If the boundary $\partial U$ does not contain any complex analytic variety of dimension one then $\partial U$ contains a “source”, that is such a point $p \in \partial U$ that $B(p, \delta) \cap U$ is a universal set for any $\delta > 0$.

**Corollary 7.4.** If $n = 1$ then any universal domain has a “source” on the boundary.

The proof of the Theorem is based on a generalization for several complex variables of a one-dimensional Hurwitz theorem. For completeness, we include the proof.

**Lemma 7.5.** Let $G, D$ be bounded domains in $\mathbb{C}^n$. Suppose that there is a sequence of domains $\{V_k\}$, $V_k \subset G$, and maps $F_k : V_k \to D$ such that

1. $F_k(V_k) = D$ and $F_k$ is a biholomorphic mapping.
2. For any compact $K \subset G$
   a) there exists a number $m$ such that $V_s \supset K$, $s \geq m$, and
   b) the sequence $\{F_s\}$ for $s \geq m$ tends uniformly on $K$ to a map $F : K \to \mathbb{C}^n$.

If $F(G)$ contains a point $z_0 \in D$ then $F$ is a biholomorphism between $G$ and $D$.

**Proof.**

1. Evidently $F(G) \subset \overline{D}$. We want to show that $F(G) \subset D$. Let $w_0 \in G$ be such a point that $F(w_0) = z_0$ and $\epsilon > 0$ be so small that the balls in Kobayashi’s metric $B_1 = B_D(z_0, \epsilon) \subset D$ and $B_2 = B_G(w_0, 2\epsilon) \subset G$. Let $z \in B_1$. For a large enough $k$, $F_k(w_0) \in B_1$. Therefore $F_k^{-1}(z) \in B_2$. Let $w$ be the limit point of $\{F_k^{-1}(z)\}$. Evidently $F(w) = z$. We have proved that $F(G) \supset B_1$. $F$ is a limit of regular holomorphic mappings. Since $G$ is connected $F$ can be either regular at every point in $G$ or the Jacobian of $F$ vanishes on $G$. In the latter case $F(G)$ could not contain any open set (by Sard’s theorem). Since $F(G) \supset B_1$, $F$ is regular on $G$. This implies that $F$ is an open mapping, so $F(G) \subset D$.

2. We will show now that $F$ is one-to-one. Let $w', w'' \in G$. For a large number $k$ and the Kobayashi metric $\rho$ we have

$$\rho_G(w', w'') = \rho_G(F_k^{-1} \circ F_k(w'), F_k^{-1} \circ F_k(w''))$$

$$\leq \rho_D(F_k(w'), F_k(w''))$$

$$\leq \rho_D(F_k(w'), F(w')) + \rho_D(F(w'), F(w'')) + \rho_D(F(w''), F_k(w'')).$$
When \( k \to \infty \) we obtain \( \rho_G(w', w'') \leq \rho_D(F(w'), F(w'')) \). Hence, if \( F(w') = F(w'') \) then \( w' = w'' \).

3. To finish the proof we have to show now that \( F(G) \supset D \). Without any loss of generality we may assume, passing to a subsequence if necessary, that \( \{F_k^{-1}\} \) converges uniformly on compacta to \( f : D \to \overline{G} \). Repeating the first step of the proof for this mapping we obtain \( f(D) \subset G \). For the mapping \( F \circ f : D \to D \) and any \( z \in D \) we have

\[
F \circ f(z) = \lim_{k \to \infty} [F_k \circ F_k^{-1}(z)] = z.
\]

Hence \( F(G) \supset D \) completing the proof of the Lemma. \( \square \)

Proof of the Theorem 7.3: In [18] the existence of two universal holomorphically non-equivalent domains is proved. Let \( G \) be one of them that is not biholomorphically equivalent to \( U \). \( G \) can be represented as \( G = \bigcup_{k=1}^{\infty} V_k \) where open sets \( V_k \subset V_{k+1} \subset \subset G \) for all \( k \). Since \( U \) is universal there exists a sequence of biholomorphic imbeddings \( f_k : U \to G \) such that \( f_k(U) \supset V_k \). Consider now \( U_k = f_k(U) \) and \( F_k = f_k^{-1} \). Since \( \{F_k\} \) is a sequence of bounded holomorphic maps we may assume, taking a subsequence if necessary, that \( \{F_k\} \) converges uniformly on any compact \( K \subset G \). Let \( F = \lim F_k \), \( F : G \to \overline{U} \). Since \( U \) is not equivalent to \( G \), \( F : G \to \partial U \) according to the lemma. Since \( \partial U \) does not contain any analytic curves, \( F(G) = p \) is a point on \( \partial U \). Let \( \delta > 0 \). We are going to prove that \( U' = B(p, \delta) \cap U \) is a universal set. Let \( M \) be any domain in \( \mathbb{C}^n \), \( K \) a compact in \( M \). Since \( G \) is universal, there exists a biholomorphic imbedding \( g : G \to M \) such that \( g(G) \supset K \). Denote \( K_1 = g^{-1}(K) \) - compact in \( G \). \( \{F_k\} \) converges uniformly on \( K_1 \) to \( p \). Therefore there exists a number \( N \) such that \( F_N(K_1) \subset U' \). Consider now \( f_N = F_N^{-1} : U' \to G \) and \( h = g \circ f_N : U' \to M \). According to the construction \( h(U'') \supset K \). This completes the proof of the Theorem. \( \square \)

Using the same Lemma we now prove the Theorem 7.1.

Proof. Pick two points: \( z_0 \in G \), \( w_0 \in D \). Since \( G \) is homogeneous then we may assume that for all \( k \), \( f_k(z_0) = w_0 \). Now, one can see that the sequence \( \{f_k\} \) converges uniformly on compacta and its limit is \( f : G \to D \) a biholomorphism. \( \square \)

### 7.2. Upper semicontinuity of automorphism groups.

It is a general geometric observation that “small perturbations can destroy symmetry but not create symmetry”. In case of domains in \( \mathbb{C}^n \) one may interpret this statement more precisely. Let \( \{D_k\}_{k=1}^{\infty} \) be bounded domains in \( \mathbb{C}^n \) and this sequence tends to a bounded domain \( D \subset \mathbb{C}^n \) in some topology on domains in \( \mathbb{C}^n \). Loosely put, is it possible that \( \text{Aut}(D_k) \) is “larger” than \( \text{Aut}(D) \) for all \( k \) large enough? If that was possible, perturbation (in the given topology) of \( D \) of any small size can create domains with more symmetry. This question is referred to as non-semi-continuity property for automorphism groups.

We will mention here several statements and examples pertaining to the question; for more detailed discussion on semi-continuity and open questions see [22, 23, 24, 40].

In the early eighties R. Greene and S.G. Krantz ([31, 32, 33]) examined this question. They proved an upper semicontinuity result in the \( C^2 \) topology, and gave the first counterexample to the upper semicontinuity principle. In [23, 24, 42] the semicontinuity question has been examined further for various other topologies. We also note here that in Riemannian geometry results of this kind have been obtained by various authors.

Here’s the statement in the \( C^\infty \) topology: the semi-continuity holds in this case; for a more precise statement see [40, p. 232].
Theorem 7.6. (Green, Krantz) Let $U \subset \mathbb{C}^n$ be a smoothly bounded, strongly pseudoconvex domain. Let $U'$ be another smoothly bounded domain whose boundary is sufficiently close to $\partial U$ in the $C^\infty$ topology. Then $\text{Aut}(U')$ is a subgroup of $\text{Aut}(U)$.

In [32] the same authors provide the following counterexample of the failure of upper semicontinuity in the $C^{1-\epsilon}$-topology (for any $\epsilon \in (0, 1)$).

Example 7.7. There are pseudoconvex domains $\{D_j\}_{j=1}^\infty$ and $D_0$, each of which is $C^\infty$ and strongly pseudoconvex except at one point, such that $\text{Aut}(D_j) \neq \{id\}$ for all $j \geq 1$, $\text{Aut}(D_0) = \{id\}$, and $D_j \to D_0$ in the $C^{1-\epsilon}$ topology, any $\epsilon \in (0, 1)$.

The other statements and examples below are in the topology induced by the Hausdorff metric; we denote the corresponding metric space by $H^n$, space of all bounded domains in $\mathbb{C}^n$ with the metric equal to the Hausdorff distance between boundaries of domains.

Example 7.8. There is a bounded $C^\infty$ domain $D$ in $\mathbb{C}$ that is not simply connected, and such that for any neighborhood $U$ of $D$ in the Hausdorff metric there is a $C^\infty$ domain $\tilde{D}$ in $U$ such that $\text{Aut}(\tilde{D})$ is not isomorphic to any subgroup of $\text{Aut}(D)$.

Construction. $B(z_0, r)$ denotes an open disk with center $z_0$ and radius $r$. Consider an $(N + 1)$-connected domain $D = \Delta \setminus \bigcup_{s=1}^{N} \Delta_s$ where $\Delta = B(0, 1)$ is the open unit disk and all $\Delta_s$ are smaller non-intersecting closed disks, whose boundaries lie entirely in $\Delta$. For a given $1 > \epsilon > 0$ fix a positive $\epsilon_1 \leq \epsilon$ and such that the set $S = \{z \in \Delta | \text{Re}(z) > -1 + \epsilon_1 \}$ contains all $\overline{\Delta}_s$. Suppose a natural number $j > 1$ is also given. We now choose a positive $\delta$ such that $L(S) \subset B(1, 1/2^j)$ where $L$ is a Möbius transformation $L(z) = \frac{z + a}{1 + za}$, and $a = 1 - \delta$.

We observe that $L(\Delta) = \Delta$. Consider now $M = \bigcap_{k=0}^{j-1} (L(D) \cdot \exp(\frac{2\pi k}{j}i))$ (each term is a rotation of $L(D)$ by angle $\frac{2\pi k}{j}$). One can verify that by construction $\mathbb{Z}_j$ acts on $M$. We define $\tilde{D}_j = L^{-1}(M)$. Then $\tilde{D}_j \subset D$, and since $\text{Aut}(\tilde{D}_j)$ is isomorphic to $\text{Aut}(M)$, $\mathbb{Z}_j$ is isomorphic to a subgroup of $\text{Aut}(\tilde{D}_j)$. Also the difference $D \setminus \tilde{D}_j \subset \Delta \setminus S$ and therefore the Hausdorff distance between $D$ and $\tilde{D}_j$ is less than $\epsilon$. If $N \geq 2$ the group $\text{Aut}(D)$ is finite. Since $j$ could be chosen to be arbitrarily large, the statement has been proved.

Remarks. The above construction works for any finitely connected domain: in any neighborhood of this domain and any integer $j$ there is a domain whose automorphism group contains $\mathbb{Z}_j$. A similar construction can be done in $\mathbb{C}^n$ for any $n \geq 1$.

Theorem 7.9. (Fridman and Poletsky [23]) Let $M$ be any domain in $\mathbb{C}^n$. Then there exists an increasing sequence of bounded domains $M_k \subset M_{k+1} \subset M$ such that $M = \cup M_k$ and $\text{Aut}(M_k)$ contains a subgroup isomorphic to $\mathbb{Z}_{n_k}$.

This statement can be proved by using the remark in the first sub-section: for each $n \in \mathbb{N}$ there exists a universal domain whose automorphism group has a subgroup isomorphic to $\mathbb{Z}_n$.

So, for any domain (even a rigid one, i.e. with $\text{Aut}(D) = \{id\}$) one can make a perturbation of less than a given size and obtain a domain with a large cyclic group. The natural question arises: which Lie groups will a similar statement hold for? It will hold for any finite group:
Theorem 7.10. Let $G$ be a group of order $m < \infty$. For any $n \geq m$ the set of bounded domains in $\mathbb{C}^n$ whose automorphism group contains a subgroup isomorphic to $G$ is everywhere dense in $H^n$.

A detailed proof of this theorem is given in [22].

So arbitrarily small perturbation of a domain in $\mathbb{C}^n$ may create a domain with a larger automorphism group. But in provided examples the groups are discrete, of dimension zero. The natural question arises: can small perturbation in $H^n$ create domains with larger dimensions of automorphism groups? The following answer is “no”.

Theorem 7.11. (Fridman, Ma, Poletsky [24]). The function $\dim(\text{Aut}(D))$ is upper semi-continuous on $H^n$.

References


fridman@math.wichita.edu, Department of Mathematics, Wichita State University, Wichita, KS 67260-0033, USA

dma@math.wichita.edu, Department of Mathematics, Wichita State University, Wichita, KS 67260-0033, USA