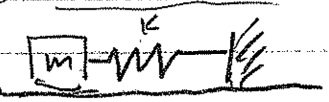
Systems of 1st order D.E.'s

ex **harmonic oscillator** $m\ddot{x}(t) + kx(t) = 0$



I.C.'s $x(t_0) = x_0$ $\quad \dot{} = \frac{d}{dt}$

$\dot{x}(t_0) = x_1$

mass-spring on frictionless table

m = mass, k = spring constant, $m, k > 0$

Newton's law $F = ma = m\ddot{x}$ plus Hooke's law $F = -kx$

(stress is proportional to strain)

lead to eq. of motion: $\ddot{x} = -\frac{k}{m}x$

Recall general soln: $x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{k}{m}}\,t\right)$

(choose A, B to satisfy I.C.'s)

Convert the 2nd order eq in one unknown to 1st order 2×2 system (linear, in this case)

$$\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

$$\dot{\underline{y}}(t) = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} y_2 \\ -\frac{k}{m}y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= A\,\underline{y} =: \underline{f}(t, \underline{y})$$

I.C.'s $\underline{y}(t_0) = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$

Recall if $A\underline{x}_k = \lambda_k \underline{x}_k$, $\lambda_k, \underline{x}_k$ e'value/e'vector of A.

Then $\underline{y}_k(t) := \underline{x}_k e^{\lambda_k t}$ satisfies $\dot{\underline{y}}_k = A\underline{y}_k$

(check: $\dot{\underline{y}}_k(t) = \lambda_k \underline{x}_k e^{\lambda_k t} = A\underline{x}_k e^{\lambda_k t} = A\underline{y}_k(t)$.)

For $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$ e'values are

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ \frac{k}{m} & \lambda \end{vmatrix} = \lambda^2 + \frac{k}{m} = 0$$

$\lambda = \pm i\sqrt{\frac{k}{m}}$. Let $\lambda_1 = i\sqrt{\frac{k}{m}}$, $\lambda_2 = -i\sqrt{\frac{k}{m}}$

e'vectors $\begin{bmatrix} \pm i\sqrt{\frac{k}{m}} & -1 \\ \frac{k}{m} & \pm i\sqrt{\frac{k}{m}} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \pm x_1 i\sqrt{\frac{k}{m}} - x_2 \\ \frac{k}{m}x_1 \pm i\sqrt{\frac{k}{m}}x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 = 1,\ x_2 = \pm i\sqrt{\frac{k}{m}}$ $\quad \underline{x}$

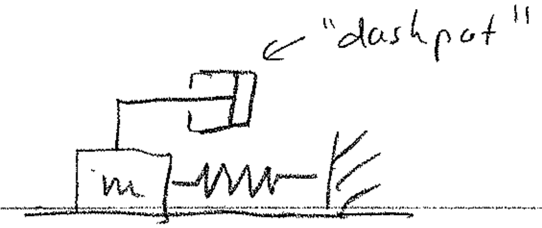
Take $\underline{x}_1 = \begin{bmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{bmatrix}$, $\underline{x}_2 = \begin{pmatrix} 1 \\ -i\sqrt{\frac{k}{m}} \end{pmatrix}$

$$\underline{y}_1(t) = \begin{bmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{bmatrix} e^{i\sqrt{\frac{k}{m}}t}, \quad \underline{y}_2(t) = \begin{bmatrix} 1 \\ -i\sqrt{\frac{k}{m}} \end{bmatrix} e^{-i\sqrt{\frac{k}{m}}t}$$

(check $A\begin{bmatrix} 1 \\ \pm i\sqrt{\frac{k}{m}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \pm i\sqrt{\frac{k}{m}} \end{bmatrix} = \begin{bmatrix} \pm i\sqrt{\frac{k}{m}} \\ -\frac{k}{m} \end{bmatrix} = \pm i\sqrt{\frac{k}{m}}\begin{bmatrix} 1 \\ \pm i\sqrt{\frac{k}{m}} \end{bmatrix}$)

(See 555 text for complete discussion)

Add damping term $c\dot{x}$, $c>0$



$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x$$

Recall soln: subst $x = e^{\lambda t}$

$$m\ddot{x} + c\dot{x} + kx = (mr^2 + cr + k)e^{rt} = 0$$

$$\therefore \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm i\mu$$

Assume $c^2 - 4mk < 0$ and $\mu = \frac{\sqrt{4mk - c^2}}{2m}$

Then soln $x(t) = e^{-\frac{c}{2m}t}(A\cos\mu t + B\sin\mu t)$,

oscillations are damped as t increase

Convert to system

$$\dot{\underline{y}}(t) = \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ -\frac{k}{m}x - \frac{c}{m}\dot{x} \end{bmatrix}$$

$$= \begin{bmatrix} y_2(t) \\ -\frac{k}{m}y_1(t) - \frac{c}{m}y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= A\underline{y}$$

e'value of A: $\begin{vmatrix} \lambda & -1 \\ \frac{k}{m} & \lambda + \frac{c}{m} \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$

so, as above, $\lambda_\pm = -\frac{c}{2m} \pm i\mu$

and we will have damping i.e. $c^2 - 4mk < 0$.