5/6/08 R-K (single step, multi stage methods) ①

e.g. 1-step, 2-stage

$s_1 = f(t_m, y_m)$

$s_2 = f(t_m + \alpha h, y_m + \beta h s_1)$

$y_{m+1} = y_m + h(a s_1 + b s_2)$

Goal: Choose α, β, a, b so that local truncation error is $O(h^3)$

Taylor series for soln $y(t)$ at $t = t_m \Rightarrow$

$$y(t_{m+1}) = y(t_m) + y'(t_m)h + y''(t_m)\frac{h^2}{2} + O(h^3)$$

Recall $y'(t_m) = f(t_m, y(t_m)) = f$

$$y''(t_m) = \frac{d}{dt} y'(t_m)$$

$$= \frac{\partial f}{\partial t}(t_m, y(t_m)) + \frac{\partial f}{\partial y} \cdot y'(t_m)$$

$$= f_t + f_y \cdot f \longleftarrow \text{(eval at } (t_m, y(t_m)))$$

$$\therefore y(t_{m+1}) = y(t_m) + fh + (f_t + f_y f)\frac{h^2}{2} + O(h^3)$$

Next note

Taylor series

$$s_2 = f(t_m + \alpha h, y_m + \beta h \overset{f}{s_1})$$

$$= f + \alpha h f_t + \beta h f \cdot f_y + O(h^2)$$

$$\therefore\ y_{n+1} = y_n + h(a s_1 + b s_2)$$

$$= y_n + h\left[af + b\left(f + \alpha h f_t + \beta h f \cdot f_y + O(h^2)\right)\right]$$

$$= y_n + (a+b) f \cdot h + b(\alpha f_t + \beta f \cdot f_y) h^2 + O(h^3)$$

For local truncation error to be $O(h^3)$
we need (with $y = y(t)$ s.t. $y(t_n) = y_n$)

$$O(h^3) = y(t_{n+1}) - y_{n+1}$$

$$= y(t_n) + fh + (f_t + f_y f)\frac{h^2}{2} + O(h^3)$$

$$- y_n - (a+b) fh - b(\alpha f_t + \beta f \cdot f_y) h^2 + O(h^3)$$

$$= \underbrace{\left[1 - (a+b)\right]}_{=0} fh + \underbrace{\left[\left(\tfrac{1}{2} - \alpha b\right) f_t + \left(\tfrac{1}{2} - \beta b\right) f \cdot f_y\right]}_{=0} h^2 + O(h^3)$$

$$\Rightarrow \quad \left.\begin{aligned} a + b &= 1 \\ \alpha b &= \tfrac{1}{2} \\ \beta b &= \tfrac{1}{2} \end{aligned}\right\} \quad \boxed{\begin{array}{c}\text{gives a} \\ \text{method of} \\ \text{order } p = 2 \\ h^{p+1} = h^3\end{array}}$$

$$\Rightarrow \quad \alpha = \beta = \frac{1}{2b}, \quad a + b = 1 \quad \Leftarrow \text{infinite \# of sol'ns}$$

One choice $a = b = \frac{1}{2}$, $\alpha = \beta = \frac{1}{2b} = 1$

i.e. $s_1 = f(t_n, y_n)$

$s_2 = f(t_n + h, y_n + h s_1)$

$y_{n+1} = y_n + h \frac{s_1 + s_2}{2}$

(trapezoid analogue, p. 192)