ODEs ($\equiv$ Ordinary Differential Equations)

Notes from A. Iserles "A First Course in the Numerical Analysis of Differential Equations"

IVP ($\equiv$ Initial Value Problem) for a 1^{st} order system

Find $y = y(t)$ for given f, y_0 s.t.

$$\boxed{\begin{aligned} y' &= f(t, y) &&\leftarrow \text{D.E.} \\ y(t_0) &= y_0 &&\leftarrow \text{I.C. = Initial condition} \end{aligned}}$$

where $t \geq t_0$, $y, y_0, f \in \mathbb{R}^d$ and $' = \frac{d}{dt}$.

(Note: We'll write y or $\underset{\sim}{y}$ for boldface vectors y. We'll try to stay close to notation in text.)

Recall: Higher order ODEs can be reduced to 1^{st} order systems.

$\|\cdot\|$ = vector norm on $\mathbb{R}^d$ — see appendix, e.g. $\|\cdot\| = \|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty, \ldots$

Def: $f = f(t, y)$ satisfies a uniform Lipschitz condition with respect to y if there is a constant λ (= Lipschitz constant) s.t.

$$\|f(t,x) - f(t,y)\| \leq \lambda \|x - y\| \text{ for all } x, y \in \mathbb{R}^d,\ t \in [t_0, t_0 + t^*].$$

E.g. If $\frac{\partial f}{\partial y}$ is bounded, we may take $\lambda = \sup \left\|\frac{\partial f}{\partial y}\right\|$

Thm (Existence and uniqueness)

Let $f(t, y)$ be continuous with respect to t and uniformly Lipschitz continuous w.r.t. y. Then there exists a unique differentiable function $y(t)$ satisfying the IVP for $t \in [t_0, t_0 + t^*]$.

Remarks

- Standard proof by Picard iteration can be turned into numerical method
- Other results are possible, e.g. f anal. in $(t, y) \Rightarrow y$ analytic.

Euler's method

Note $f(t, y(t)) \approx f(t_0, y(t_0))$ $\quad t \in [t_0, t_0+h]$, $h > 0$ small.

Then
$$y(t) = y(t_0) + \int_{t_0}^{t} \underbrace{f(\tau, y(\tau))}_{y'(t)}\, d\tau$$
$$\approx y_0 + (t - t_0)\, f(t_0, y_0)$$

Given t_0, $t_1 = t_0 + h$, $t_2 = t_0 + 2h, \ldots$

where $h > 0$ is the (small) <u>time step</u>

we denote
$$y_n \approx y(t_n) \qquad n = 0, 1, 2, \ldots$$
y_n: numerical approximate; $y(t_n)$: exact solution

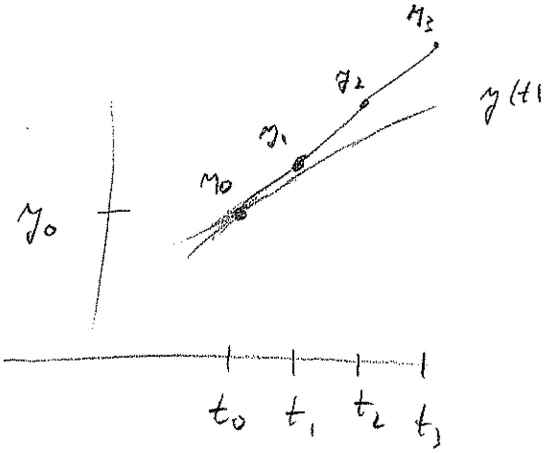
where
$$\boxed{\begin{aligned} y_1 &= y_0 + h\, f(t_0, y_0) \\ &\;\vdots \\ y_{n+1} &= y_n + h\, f(t_n, y_n) \end{aligned}} \Bigg\} \text{ Euler's method}$$

y_{n+1} is given explicitly

picture

$(d=1)$

(see text for better picture)



} error

Q: How does error behave as e.g. $h \to 0$?

Main methods ~~with~~:
- Multistep methods
- Runge-Kutta methods

} improvements on Euler

Variable time step $\quad h_n = t_{n+1} - t_n$

$$y_{n+1} = y_n + h_n f(t_n, y_n)$$

Many good software packages use variable time-stepping.

<u>Convergence</u> in $[t_0, t_0 + t^*]$

as $h \to 0 \quad n = 0, 1, \ldots, \lfloor t^*/h \rfloor$ = greatest integer $\leq t^*/h$.

Let $y_n = y_{n,h}$. A method is <u>convergent</u>

if for every $t^* > 0$

$$\lim_{h \to 0+} \max_{n=0,1,\ldots,\lfloor t^*/h \rfloor} \| y_{n,h} - y(t_n) \| = 0$$

i.e. the numerical solution converges to the true solution on the grid $t_0, t_1, \ldots, t_n, \ldots$

(<u>Note</u>: Since all vector <u>norms</u> $\|\cdot\|$ are <u>equivalent</u>, it doesn't matter which we use.
(751 students?))

Thm 1.1 Euler's method is convergent.

Pf: Assume f (and $\therefore y$) are smooth enough to use Taylor's Thm.

Let $e_{n,h} := y_{n,h} - y(t_n)$ = numerical error

want to prove $\lim_{h \to 0+} \max_n \| e_{n,h} \| = 0$

Taylor's Thm $\Rightarrow$

$$y(t_{n+1}) = y(t_n) + h\, y'(t_n) + O(h^2)$$
$$= y(t_n) + h\, f(t_n, y(t_n)) + O(h^2)$$

Recall

$$y_{n+1,h} = y_{n,h} + h\, f(t_n, y_{n,h})$$

$\therefore$

$$e_{n+1,h} = y_{n+1,h} - y(t_{n+1})$$
$$= y_{n,h} - y(t_n) + h\left[f(t_n, y_{n,h}) - f(t_n, y(t_n))\right] + O(h^2)$$
$$= e_{n,h} + h\left[f(t_n, y(t_n) + e_{n,h}) - f(t_n, y(t_n))\right] + O(h^2)$$

And so by Lipschitz cond. on f $\quad \| f(t_n, y(t_n) + e_{n,h}) - f(t_n, y(t_n)) \| \le \lambda \| e_{n,h} \|$

$$\| e_{n+1,h} \| \le \| e_{n,h} \| + h\lambda \| e_{n,h} \| + O(h^2)$$

i.e. $\boxed{\| e_{n+1,h} \| \le (1 + h\lambda) \| e_{n,h} \| + c h^2}$ for some $c > 0$

$$n = 0, 1, \ldots, \lfloor t^*/h \rfloor - 1$$

e.g. $c = \max_{t \in [t_0, t_0 + t^*]} \| y''(t) \|$

Claim $\boxed{\|e_{n,h}\| \leq \frac{c}{\lambda} h\left[(1+h\lambda)^n - 1\right]}$ $\quad n = 0, 1, \ldots$

Proof by induction on n

$\underline{n=0}$ $\quad \|e_{0,h}\| \leq 0 \quad$ since $e_{0,h} = y_{0,h} - y(t_0) = 0$ by I.C.

Induction step $\quad$ Assume true for n + show true for $n+1$

$$\|e_{n+1,h}\| \leq (1+h\lambda)\|e_{n,h}\| + ch^2 \qquad \text{by above}$$

$$\leq (1+h\lambda)\frac{c}{\lambda}h\left[(1+h\lambda)^n - 1\right] + ch^2$$

$$= \frac{c}{\lambda}h\left[(1+h\lambda)^{n+1} - (1+h\lambda)\right] + \frac{c}{\lambda}h(h\lambda)$$

$$= \frac{c}{\lambda}h\left[(1+h\lambda)^{n+1} - 1\right]$$

$\therefore$ result is true

Next, note $\quad 1 + h\lambda < e^{h\lambda} \quad$ since $h\lambda > 0$

$\therefore \quad (1+h\lambda)^n < e^{nh\lambda} < e^{t^*\lambda} \quad$ for $n = 0, 1, \ldots, \lfloor t^*/h \rfloor$

And so $\quad \boxed{\|e_{n,h}\| \leq \underbrace{\frac{c}{\lambda}\left(e^{t^*\lambda} - 1\right)}_{\text{independent of } h} h} \rightarrow 0, \; h \rightarrow 0$

$\forall\, n = 0, 1, \ldots, \lfloor t^*/h \rfloor$

Note: This may be a great overestimate of the actual error as the text example shows

+ So $\quad \lim\limits_{\substack{h \to 0 \\ 0 \leq nh \leq t^*}} \|e_{n,h}\| = 0$

qed

Order of accuracy

Euler's method: $y_{n+1} - [y_n + hf(t_n, y_n)] = 0$

replace y_n, y_{n+1} by exact soln $y(t_n), y(t_{n+1})$
+ use Taylor series

$$y(t_{n+1}) - [y(t_n) + hf(t_n, y(t_n))]$$

$$= \left(y(t_n) + h y'(t_n) + O(h^2)\right) - \left[y(t_n) + h y'(t_n)\right]$$

$$= O(h^2) \quad = \text{local error}$$

$$h^2 = h^{p+1}, \quad p = 1$$

Euler's method is of order 1

more generally given time stepping scheme

$$y_{n+1} = \mathcal{Y}_n(f, h, y_0, y_1, \ldots, y_n), \quad n = 0, 1, \ldots$$

if for suff. smooth f + $y(t)$ = exact soln.

$$y(t_{n+1}) - \mathcal{Y}_n(f, h, y(t_0), y(t_1), \ldots, y(t_n)) = O(h^{p+1})$$

= local error

The scheme is said to of order p.

(more general def + criterion later
Note: def is indep. of f)

Idea

$$\overbrace{\text{local error}}^{\text{truncation or discretization}} = O(h^{p+1})$$

$\times$ $\qquad\qquad\qquad\qquad$ $\times$

$\#$ of time steps = $O\left(\frac{1}{h}\right)$
to get from t_0 to $t_0 + t^*$

Final error = $O(h^p)$

if method converges!

as in case of Euler's method

1.3 <u>The trapezoidal rule</u> $\quad y' = f(t, y)$

$$y(t) = y(t_n) + \int_{t_n}^{t} f(\tau, y(\tau))\, d\tau$$

$$\approx y(t_n) + \tfrac{1}{2}(t - t_n)\left\{ f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})) \right\}$$

<u>trapezoidal rule</u> $\quad y_{n+1} = y_n + \frac{1}{2} h \left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right]$

y_{n+1} is given <u>implicitly</u> and ~~must~~
a nonlinear eq. must be solved at each step.

<u>order</u> – substitute exact soln if suff. smooth:

$$y(t_{n+1}) - \left(y(t_n) + \tfrac{1}{2} h \left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1})) \right] \right)$$

$$= y(t_{n+1}) - \left(y(t_n) + \tfrac{1}{2} h \left[y'(t_n) + y'(t_{n+1}) \right] \right)$$

$$= y(t_n) + h y'(t_n) + \tfrac{1}{2} h^2 y''(t_n) + O(h^3)$$
$$- \left(y(t_n) + \tfrac{1}{2} h \left[y'(t_n) + y'(t_n) + h y''(t_n) + O(h^2) \right] \right)$$

$$= O(h^3)$$

$\therefore$ trapezoidal rule is order 2.

Thm 1.2 The trapezoidal rule is convergent.

Pf:

$$e_{n+1,h} = y_{n+1} - y(t_{n+1})$$

$$= y_n + \frac{1}{2}h\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

$$-\left(y(t_n) + \frac{1}{2}\left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right]\right) + O(h^3)$$

$$= e_{n,h} + \frac{1}{2}h\Big[\big(f(t_n, y_n) - f(t_n, y(t_n))\big)$$

$$+ f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y(t_{n+1}))\Big] + O(h^3)$$

$$\therefore \|e_{n+1,h}\| \le \|e_{n,h}\| + \frac{1}{2}h\lambda\left(\|e_{n,h}\| + \|e_{n+1,h}\|\right) + ch^3$$

e.g. $c \approx \sup \|y'''\|$,

Since $h \to 0$ assume $0 < h\lambda < 2$

Then

$$\left(1 - \frac{1}{2}h\lambda\right)\|e_{n+1,h}\| \le \left(1 + \frac{1}{2}h\lambda\right)\|e_{n,h}\| + ch^3$$

or

$$\|e_{n+1,h}\| \le \left(\frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda}\right)\|e_{n,h}\| + \frac{c}{1 - \frac{1}{2}h\lambda}h^3$$

Again by induction on n

$$\|e_{n+1,h}\| \le \frac{c}{\lambda}\left[\left(\frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda}\right)^n - 1\right]h^2 \qquad (1.11)$$

$\underline{n=0}$ $\quad e_{0,h} = y_0 - y(t_0) = 0$

<u>induction step</u>

Assume $\quad \|e_{n,h}\| \le \frac{c}{\lambda}\left(\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^n - 1\right)h^2$

$$\|e_{n+1,h}\| \le \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)\|e_{n,h}\| + \left(\frac{c}{1-\frac{1}{2}h\lambda}\right)h^3$$

$$\le \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)\frac{c}{\lambda}\left(\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^n - 1\right)h^2 + \left(\frac{c}{1-\frac{1}{2}h\lambda}\right)h^3$$

$$= \frac{c}{\lambda}\left(\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^{n+1} - \frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)h^2 + \frac{c}{\lambda}\left(\frac{h\lambda}{1-\frac{1}{2}h\lambda}\right)h^2$$

$$= \frac{c}{\lambda}\left(\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^{n+1} - \frac{1-\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)h^2$$

$$= \frac{c}{\lambda}\left(\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^{n+1} - 1\right)h^2$$

Qed

$$\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda} = 1 + \frac{h\lambda}{1-\frac{1}{2}h\lambda} \le \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{h\lambda}{1-\frac{1}{2}h\lambda}\right)^{l} = \exp\left(\frac{h\lambda}{1-\frac{1}{2}h\lambda}\right)$$

$$\therefore \quad \|e_{m,h}\| \le \frac{ch^2}{\lambda}\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^{m} \le \frac{ch^2}{\lambda}\exp\left(\frac{mh\lambda}{1-\frac{1}{2}h\lambda}\right)$$

$$\therefore \quad mh \le t^* \Rightarrow \|e_{m,h}\| \le \underbrace{\frac{c}{\lambda}\exp\left(\frac{t^*\lambda}{1-\frac{1}{2}h\lambda}\right)}_{\text{may be very large! overestimate}} h^2 = \underline{\underline{O(h^2)}}$$

$p = 2$

$$\text{and} \quad \lim_{\substack{h\to 0 \\ 0\le mh\le t^*}} \|e_{m,h}\| = 0$$

(QED

Note we must solve nonlinear eq.s

$$y_{m+1} - \frac{1}{2}h\, f(t_{m+1}, y_{m+1}) = y_m + \frac{1}{2}h\, f(t_m, y_m)$$

for y_{m+1} at each step. (Use iterative Newton solver with $y_{m+1}^{(0)} = y_m$ as initial guess.)

- Second order accuracy is gained at higher cost.
- Usually evaluating $f(t, y)$ is most expensive computation since f may be a complicated function