Math 551 - DeLillo
Homework for Chap 6
due Th 5/1/08

Written HW

(8.) Do 6.20 from text

Computer HW VI

Program Simpson's rule. You may revise my trapzd2(f, a, b, n). Compare your program with trapzd2 and quadtx for some selected integrals, e.g. $\int_0^1 f(x)\,dx$ where $f(x) = x^2, x^3,$ and x^4. Also try the integral for π in 6.3 and 6.4.

An alternate calculation of the NC weights

see P 4.1.2 (2^{nd} edition) = P 4.1.4 (1^{st} edition)

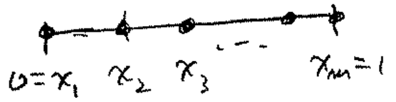
Recall the m-pt closed NC rules should compute $\int_0^1 x^{i-1}\,dx = \frac{x^i}{i}\Big|_0^1 = \frac{1}{i}$ exactly

$(i = 1:m)$

i.e $Q_{NC(m)} = \sum_{k=1}^{m} w_k f_k = \sum_{k=1}^{m} w_k x_k^{i-1} = \frac{1}{i}$

$x_k := \frac{k-1}{m-1},\ k = 1:m)$

x_k's equidist.



or

$$
\begin{array}{ll}
i=1 & w_1 + w_2 + \ldots + w_m = 1 \\
i=2 & w_1 x_1 + w_2 x_2 + \ldots + w_m x_m = \frac{1}{2} \\
i=3 & w_1 x_1^2 + w_2 x_2^2 + \ldots + w_m x_m^2 = \frac{1}{3} \\
\vdots & \vdots \\
i=m & w_1 x_1^{m-1} + w_2 x_2^{m-1} + \ldots + w_m x_m^{m-1} = \frac{1}{m}
\end{array}
$$

$\downarrow$ Think matrix-vector

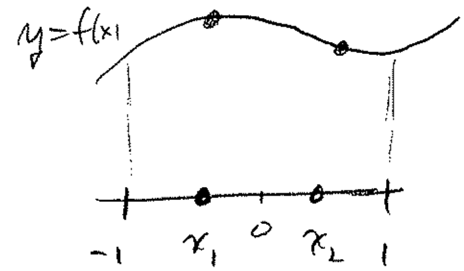
$$
A w = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_m \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_m^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{m-1} & x_2^{m-1} & x_3^{m-1} & \cdots & x_m^{m-1} \end{bmatrix}}_{} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ \vdots \\ 1/m \end{bmatrix} = r
$$

A = Vandermonde matrix again! since x_k's are given.

MATLAB soln: $\boxed{w = A \backslash r}$

(Notes from van Loan's book)

4.1.4 Gauss Quadrature



Problem Determine w_1, w_2, x_1, x_2 such that

$$w_1 f(x_1) + w_2 f(x_2) = \int_{-1}^{1} f(x)\,dx$$

$x_1, x_2 \in [-1, 1]$

exact!

for $f(x) =$ polynomial of degree ≤ 3.

degree		
0	$f(x)=1$	$\Rightarrow w_1 + w_2 = \int_{-1}^{1} 1 \cdot dx = 2$
1	$f(x)=x$	$\Rightarrow w_1 x_1 + w_2 x_2 = \int_{-1}^{1} x\,dx = 0$
2	$f(x)=x^2$	$\Rightarrow w_1 x_1^2 + w_2 x_2^2 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big\|_{-1}^{1} = \frac{2}{3}$
3	$f(x)=x^3$	$\Rightarrow w_1 x_1^3 + w_2 x_2^3 = \int_{-1}^{1} x^3 dx = 0$

i.e.

$$\left.\begin{aligned} w_1 + w_2 &= 2 \\ w_1 x_1 + w_2 x_2 &= 0 \\ w_1 x_1^2 + w_2 x_2^2 &= \tfrac{2}{3} \\ w_1 x_1^3 + w_2 x_2^3 &= 0 \end{aligned}\right\}$$

4 nonlinear eqs. in 4 unknowns w_1, w_2, x_1, x_2

soln: 4th eq $- x_1^2 \times$ 2nd eq $= \cancel{w_1 x_1^3} + w_2 x_2^3 - \cancel{w_1 x_1^3} - w_2 x_2 x_1^2$

$$= w_2 x_2 (x_2^2 - x_1^2) = 0$$

$$\Rightarrow x_2^2 = x_1^2 = x_2 = \pm x_1$$

$\therefore$ $\boxed{x_2 = -x_1}$ since x_1, x_2 must be distinct ($x_1 \neq x_2$)

$\underline{2^{nd} \text{ eq}} \Rightarrow \quad 0 = \omega_1 x_1 + \omega_2 x_2 = \omega_1 x_1 - \omega_2 x_1 = (\omega_1 - \omega_2) x_1$

$$\Rightarrow \boxed{\omega_1 = \omega_2}$$

$\underline{1^{st} \text{ eq}} \Rightarrow \quad 2 = \omega_1 + \omega_2 = 2\omega_1 \Rightarrow \boxed{\omega_1 = \omega_2 = 1}$

$\underline{3^{rd} \text{ eq}} \Rightarrow \quad \frac{2}{3} = \underset{1}{\overset{\shortparallel}{\omega_1}} x_1^2 + \underset{1}{\overset{\shortparallel}{\omega_2}} x_2^2 = 2 x_1^2$

$$\Rightarrow x_1^2 = \frac{1}{3} \Rightarrow \boxed{\begin{aligned} x_1 &= -\frac{1}{\sqrt{3}} \\ x_2 &= \frac{1}{\sqrt{3}} \end{aligned}}$$

i.e. $\displaystyle\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) + \omega_2 f(x_2)$

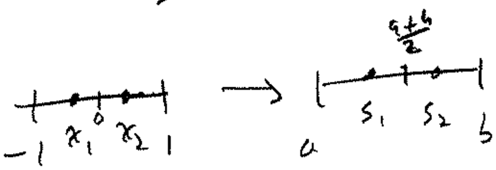
$$= \boxed{f\left(-\tfrac{1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right) =: Q_{GL(2)}}$$

2 pt Gauss-Legendre rule or (Gauss quad.)

$\underline{\text{Note}}$ For $\displaystyle\int_a^b f(s)\,ds$, we need change-of-variables

$$s = \frac{b-a}{2} x + \frac{a+b}{2}$$

taking



Then $\displaystyle\int_a^b f(s)\,ds = \frac{b-a}{2} \int_{-1}^{1} g(x)\,dx$ where $g(x) = f\left(\frac{b-a}{2} x + \frac{a+b}{2}\right)$

$$\approx \frac{b-a}{2}\left[g(x_1) + g(x_2)\right]$$

$\underline{\text{Note}}$: For any cubic $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\int_{-1}^{1} f(x)\,dx = a_0 \int_{-1}^{1} dx + a_1 \int_{-1}^{1} x\,dx + a_2 \int_{-1}^{1} x^2\,dx + a_3 \int_{-1}^{1} x^3\,dx \overset{\text{exact}}{=} f\left(\tfrac{-1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right)$$

m-pt Gauss-Legendre rule

$$Q_{GL(m)} = w_1 f(x_1) + w_2 f(x_2) + \dots + w_m f(x_m)$$

where for $f(x)$ with $x \in [-1,1]$ we have

$$w_1 f(x_1) + \dots + w_m f(x_m) \underset{\text{exact}}{=} \int_{-1}^{1} f(x)\,dx \quad (*)$$

for $f(x)$ polynomial of degree $\le 2m-1$ **

(** This is the best we can expect with $2m$ unknowns w_k, x_k $k=1:m$)

For $f(x) = 1, x, x^2, \dots, x^{2m-1}$

$$(*) \Rightarrow \quad w_1 x_1^k + w_2 x_2^k + \dots + w_m x_m^k = \int_{-1}^{1} x^k\,dx$$

$$= \left.\frac{x^{k+1}}{k+1}\right|_{-1}^{1} = \frac{1-(-1)^{k+1}}{k+1}$$

i.e.

$$\left.\begin{array}{ll}
(k=0) & w_1 + w_2 + \dots + w_m = 2 \\
(k=1) & w_1 x_1 + w_2 x_2 + \dots + w_m x_m = 0 \\
(k=2) & w_1 x_1^2 + w_2 x_2^2 + \dots + w_m x_m^2 = \frac{2}{3} \\
\vdots & \dots \\
(k=2m-1) & \underbrace{w_1 x_1^{2m-1} + w_2 w_2^{2m-1} + \dots + w_m x_m^{2m-1} = 0}_{\text{How to solve? See Math 751 text Numerical Linear Algebra Lect. 37.}}
\end{array}\right\}$$

$2m$ nonlinear eq in $2m$ unknowns w_k, x_k, $k=1:m$

Solve once + save

Error estimate

CBS $$\left| \int_a^b f(x)dx - Q_{GL(m)} \right| \le \frac{(b-a)^{2m+1}(m!)^4}{(2m+1)[(2m)!]^3} M_{2m}$$

where $|f^{(2m)}(x)| \le M_{2m}$ for $x \in [a,b]$.

Note If $f(x)$ is a poly. of degree $\le 2m-1$ then we can take $M_{2m} = 0$ and error $= 0$.

Note: We have (roughly) that $h \approx \frac{b-a}{m}$

$\therefore$ error $\approx O(h^{2m})$ — very rough since x_k's are not equally distributed.

4.4.2 Spline Quadrature

$S(x) =$ cubic spline interpolant at (x_i, y_i) $i = 1:m$

$y_i = f(x_i)$.

Then $q_i(x) = p_{i,4} + p_{i,3}(x - x_i) + p_{i,2}(x-x_i)^2 + p_{i,1}(x-x_i)^3$

integrate term-by-term

And $$\int_{x_i}^{x_{i+1}} q_i(x)dx = p_{i,4} h_i + \frac{p_{i,3}}{2} h_i^2 + \frac{p_{i,2}}{3} h_i^3 + \frac{p_{i,4}}{4} h_i^4$$

$h_i := x_{i+1} - x_i$

$$\therefore \int_a^b f(x)dx \approx \int_{x_1}^{x_m} S(x)dx = \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} q_i(x)dx$$

Section 4.5 – skip

```
% Script File: GLvsNC
%
% Compares the m-point Newton-Cotes and Gauss-Legendre rules
% applied to the integral of sin(x) from 0 to pi/2.
%
  clc
  home
  disp(' m              NC(m)                    GL(m) ')
  disp('---------------------------------------------------')
  for m=2:6
     NC = QNC('sin',0,pi/2,m);
    GL = QGL('sin',0,pi/2,m);
    disp(sprintf(' %1.0f  %20.16f   %20.16f',m,NC,GL))
  end
```

```
  function numI = QGL(fname,a,b,m)
%
% Pre:
%   fname    string that names an available function of the
%            form f(x) that is defined on [a,b]. f should
%            return a column vector if x is a column vector.
%     a,b    real scalars
%       m    integer that satisfies 2 <= m <=6
% Post:
%    numI    the m-point Gauss-Legendre approximation of the
%            integral of f(x) from a to b.
%
   [w,x] = WGL(m);
   fvals = feval(fname,((b-a)/2)*x + ((a+b)/2)*ones(m,1));
   numI = ((b-a)/2)*w'*fvals;
```

```
  function [w,x] = WGL(m);
%
% Pre:
%   m    integer that satisfies 2 <= m <= 6
%
% Post:
    w    column m-vector consisting of the weights for the
         m-point Gauss-Legendre rule..
%   x    column m-vector consisting of the abscissae for the
%        m-point Gauss-Legendre rule.
%

  w = ones(m,1);
  x = ones(m,1);

  if m==2
     w(1) =  1.000000000000000; w(2) =  w(1);
     x(1) = -0.577350269189626; x(2) = -x(1);
  elseif m==3
     w(1) =  0.555555555555558; w(3) =  w(1);
     w(2) =  0.888888888888889;
     x(1) = -0.774596669241483; x(3) = -x(1);
     x(2) =  0.000000000000000;
  elseif m==4
     w(1) =  0.347854845137454; w(4) =  w(1);
     w(2) =  0.652145154862546; w(3) =  w(2);
     x(1) = -0.861136311594053; x(4) = -x(1);
     x(2) = -0.339981043584856; x(3) = -x(2);
  elseif m==5
     w(1) =  0.236926885056189; w(5) =  w(1);
     w(2) =  0.478628670499366; w(4) =  w(2);
     w(3) =  0.568888888888889;
     x(1) = -0.906179845938644; x(5) = -x(1);
     x(2) = -0.538469310105683; x(4) = -x(2);
     x(3) =  0.000000000000000;
  else
     w(1) =  0.171324492379170; w(6) =  w(1);
     w(2) =  0.360761573048139; w(5) =  w(2);
     w(3) =  0.467913934572691; w(4) =  w(3);
     x(1) = -0.932469514203152; x(6) = -x(1);
     x(2) = -0.661209386466265; x(5) = -x(2);
     x(3) = -0.238619186083197; x(4) = -x(3);
  end;
```