4/15/08

CH. 6 - Quadrature

"numerical integration"

Fundamental Theorem of Calculus: $\int_a^b f(x)\,dx = F(b) - F(a)$

if $F'(x) = f(x)$

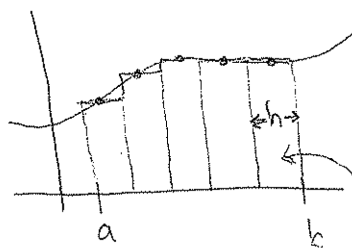
(F = anti derivative of $f(x)$)

For general $f(x)$ we cannot expect to find an antiderivative in "closed form" [i.e. as some combination of known functions]

$\therefore$ to find $\int_a^b f(x)\,dx$ we have to approximate it numerically

3 methods from Calc. I

1) mid point rule



$y=f(x)$

calculate f at midpoints

$h = \frac{b-a}{5}$

area of last box $= hf(x_5)$

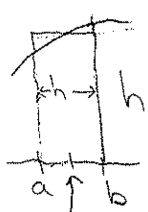
$$\int_a^b f(x)\,dx \approx h[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)]$$

$\therefore$ take n intervals $\quad h = \frac{b-a}{n}$

& midpoints $x_1, x_2, \cdots, x_n$

Then $M = h\sum_{i=1}^{n} hf(x_i)$.

Look at one "panel"



$h = b-a$

$\left(\frac{a+b}{2}\right)$

$M = hf\left(\frac{a+b}{2}\right)$

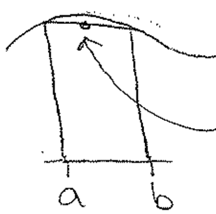
2) Trapezoidal rule

adaptive method exist in Matlab for things like this

area of trapezoid

$$T = h\frac{f(a)+f(b)}{2}$$

1 Trapezoid

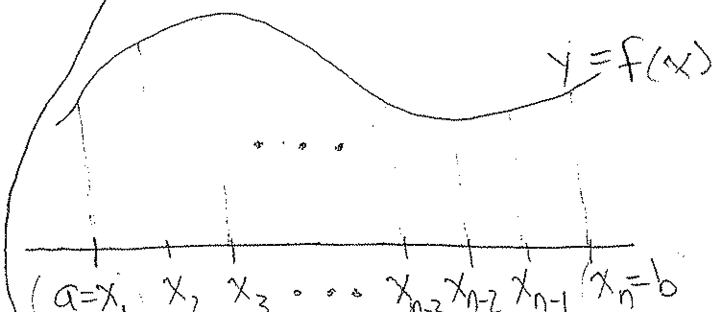


$\frac{1}{2}(f(a)+f(b))$ = aver.

$h = b-a$

$$T = h\left(\frac{f(x_1)+f(x_2)}{2} + \frac{f(x_2)+f(x_3)}{2} + \frac{f(x_3)+f(x_4)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right)$$

$$\therefore T = \frac{h}{2}\Big[f(x_1)+2f(x_2)+2f(x_3) \cdots + 2f(x_{n-1})+f(x_n)\Big]$$



$x_i = a+(i-1)h \;,\quad i=1,\dots,n$

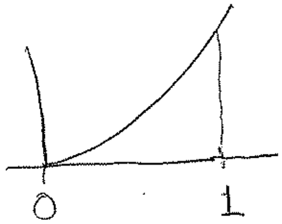
$x_1 = a$
$x_2 = a+h$
$x_3 = a+2h$
$\vdots$
$x_n = a+(n-1)h = a+b-a = b$

(using Trapezoidal Rule)
see traptd2.m in Matlab

```
>> f = inline('x.^2')

>> traptd2(f,0,1,2)

>> traptd2(f,0,1,5)
```



$f(x) = x^2$
$n = 2$

$h = \frac{1-0}{2-1} = 1$

$x_1 = a = 0$
$x_2 = a + h = 1$
$T = h/2\,(f(x_1) + f(x_2))$
$= \frac{1}{2}(f(0) + f(1))$
$= \frac{1}{2}(0 + 1^2)$
$= \frac{1}{2}$

try for larger & larger #s to get to

```
>> traptd2(f,0,1,1000000)
```

exact: $\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

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see quad.m (using Simpson's Rule)

```
>> tol = 1.e-16
```
} for better tolerance

```
>> [Q, k] = quad(f, 0, 1, tol)
```

# of func. eval.

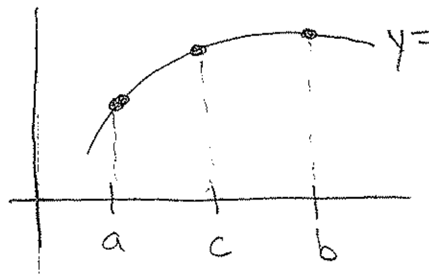
took 13 evaluations to get $\frac{1}{3}$ to 16 places
∴ Simpson's Rule worked much better than trap.

try
```
>> f = inline('x.^4')
>> [Q,k] = quad(f, 0, 1, tol)
```
"took 2049 eval's."

```
>> traptd2(f,0,1,100000)
```
"not quite"

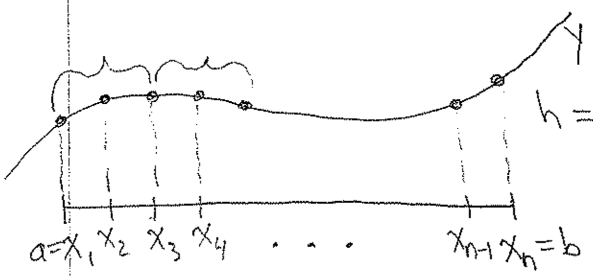
3) Simpson's Rule

$$S = \frac{h}{6}\left[f(a) + 4f(c) + f(b)\right]$$



$c = \frac{a+b}{2}$ = midpt.

$h = b - a$



$y=f(x)$

$$h = \frac{b-a}{(n-1)/2} = \frac{b-a}{k}$$

n must be of the form $n = 2k+1$

$k = 1, 2, \ldots$

Then Simpson's Rule becomes

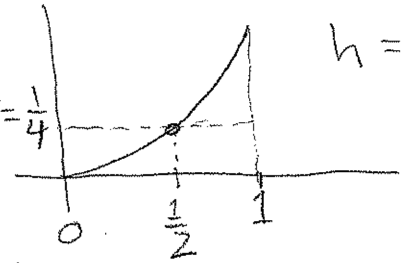
$$S = \frac{h}{6}\Big[\underbrace{f(x_1)}_{f(a)} + 4f(x_2) + 2f(x_3) + 4f(x_4) + \cdots$$

$$\cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

(see Calc. I text)

USING MIDPOINT & TRAP
TO DERIVE
SIMPSON'S RULE

$$\int_0^1 x^2\,dx = \frac{1}{3} \text{ exact}$$



$h = \overset{b\,-\,a}{1 - 0} = 1$

midpt.
$$M = h f\left(\frac{a+b}{2}\right)$$

$$= 1 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

trap.
$$T = h\,\frac{f(a) + f(b)}{2} = 1 \cdot \frac{0+1}{2} = \frac{1}{2}$$

ERROR: for $M = \frac{1}{3} - \frac{1}{4}$ is $\frac{1}{12}$

for $T = \frac{1}{3} - \frac{1}{2}$ is $-\frac{1}{6}$

error w/ T is $-2 *$ error w/ M

$$\underset{\substack{\Uparrow \\ \text{exact}}}{S} - T = -2(S - M)$$

$$3S = T + 2M$$

$$S = \frac{2}{3}M + \frac{1}{3}T$$

using M & T from above

$$S = \frac{2h}{3}\left(f\left(\frac{a+b}{2}\right)\right) + \frac{h}{3}\cdot\frac{1}{2}\left(f(a) + f(b)\right)$$

$$= \frac{h}{6}\Big[f(a) + 4f(\underset{\substack{\Uparrow \\ \frac{a+b}{2}}}{c}) + f(b)\Big]$$