2 Methods for solving nonlinear eqs $f(x)=0$ numerically

(1) Picard iteration

$$f(x) = \frac{1}{2}x^2 - x + \frac{1}{4}$$

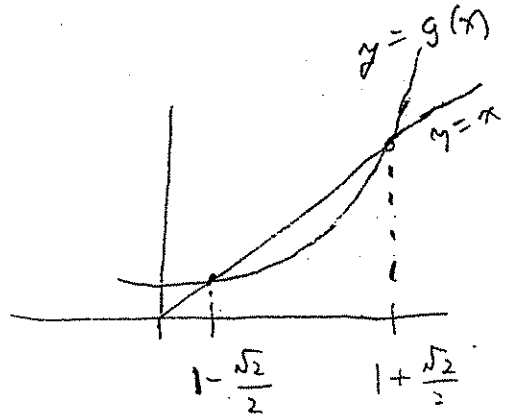
Solve $f(x) = 0$

$$x^2 - 2x + \frac{1}{2} = 0$$

$$x = \frac{2 \pm \sqrt{4-2}}{2} = 1 \pm \frac{\sqrt{2}}{2} \approx .292893219\ldots \text{ or } 1.707106781\ldots$$

Note $f(x) = g(x) - x$

where $\boxed{g(x) = \frac{1}{2}x^2 + \frac{1}{4}}$



To solve $f(x) = 0$

Find x s.t. $x = g(x)$

Picard iteration

initial guess	x_0
iteration	$x_{n+1} = g(x_n)$

(**Def**: $\lim_{n\to\infty} x_n = x^*$ if for all $\varepsilon > 0$ there exists an $N > 0$ s.t. $n > N \Rightarrow |x^* - x_n| < \varepsilon$.)

If g is continuous then

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(x_n) = g(x^*)$$

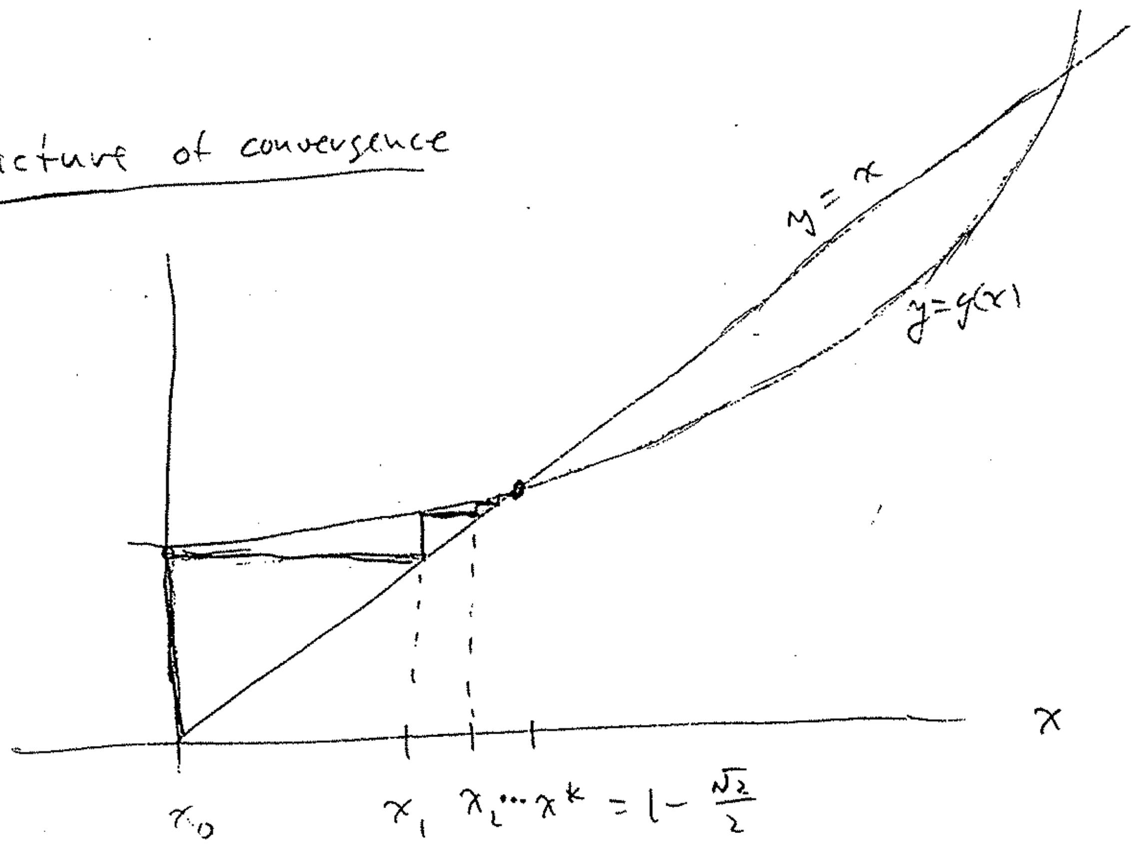
So $x^* = g(x^*)$ and x^* is the solution

The x_n's are called the iterates and are said to converge to the solution, x^*.

Picard iteration from initial guess $x_0 = 0$

n	x_n	$g(x_n) = \frac{1}{2}x_n^2 + \frac{1}{4}$
0	0	.25
1	.25	.28125
2	.28125	.289550781
3	.289550781	.291919827
4	.291919827	.292608593
5	.292608593	.292809894
6	.292809894	.292868817
⋮	⋮	⋮
	x^* =	.292893219

picture of convergence



Convergence analysis for Picard iteration

Suppose $f(x) = g(x) - x$ where $g'(x)$ exists and $f(x^*) = 0$, i.e. $x^* = g(x^*)$.

The error at the m^{th} iterate is

$$
\begin{aligned}
e_m &= x^* - x_m \\
&= g(x^*) - g(x_{m-1}) \\
&= g'(c_m)(x^* - x_{m-1}) \\
&= g'(c_m)\, e_{m-1} \\
&\;\vdots \\
&\approx [g'(x^*)]^m e_0
\end{aligned}
$$

for some c_m between x^* and x_{m-1} by the Mean Value Theorem

(linear convergence, not so fast)

since $g'(c_m) \approx g'(x^*)$ if x_m is near x^*.

That is, if we have a close enough initial guess the error will improve by a factor of about $g'(x^*)$ at each step. So if $g'(x^*) \approx \frac{1}{10}$ we would gain a decimal place each step

Note for g in our example

$$g'(x^*) = x^* = .292\ldots$$

(2) <u>Newton's method</u> for $f(x) = \frac{1}{2}x^2 - x + \frac{1}{4}$

$$f'(x) = x - 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{2}x_n^2 - x_n + \frac{1}{4}}{x_n - 1}$$

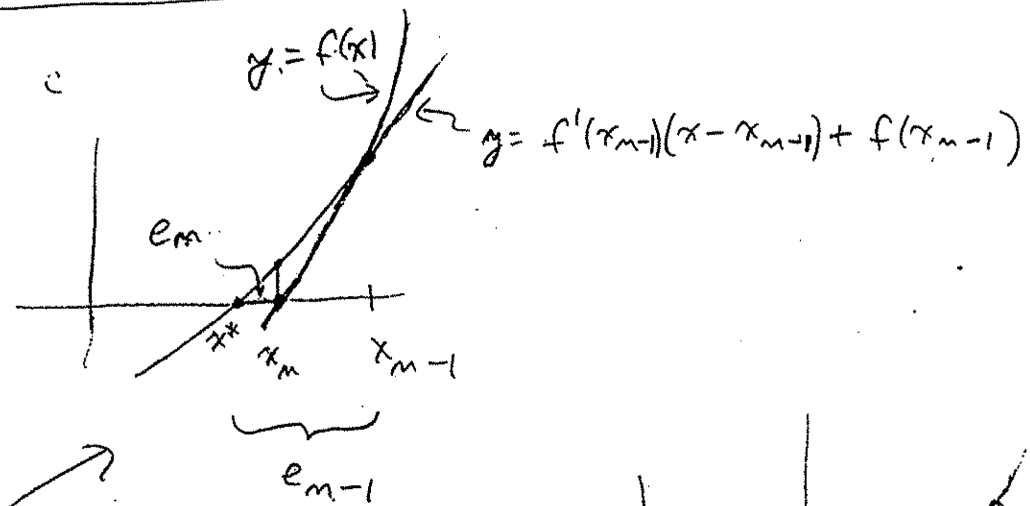
$$= \frac{x_n^2 - x_n - \frac{1}{2}x_n^2 + x_n - \frac{1}{4}}{x_n - 1}$$

$$= \frac{\frac{1}{2}x_n^2 - \frac{1}{4}}{x_n - 1} = \frac{x_n^2 - \frac{1}{2}}{2(x_n - 1)}$$

n	x_n	$x_{n+1} = \frac{x_n^2 - \frac{1}{2}}{2(x_n-1)}$
0	0	$\frac{1}{4} = .25$
1	.25	$\frac{7}{24} = .291666667$
2	.291666667	.292892157
3	.292892157	.292893219

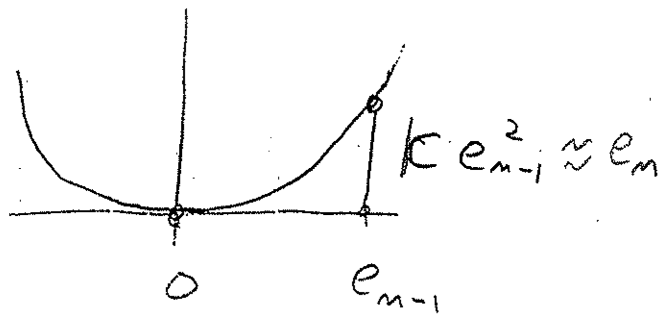
Note how much faster Newton's method converges.

Convergence analysis for Newton's method



This is like a parabola

(more in Calculus II)



<u>error at nth iteration</u> =

$$e_n = x^* - x_n$$

$$= x^* - x_{n-1} + f'(x_{n-1})^{-1} f(x_{n-1})$$

$$= f'(x_{n-1})^{-1} \left[f'(x_{n-1})(x^* - x_{n-1}) + f(x_{n-1}) \right]$$

$$= f'(x_{n-1})^{-1} \left[C \cdot (x^* - x_{n-1})^2 \right] \qquad \text{(Calc II)}$$

$$= \underbrace{f'(x_{n-1})^{-1} C}_{} \cdot e_{n-1}^2 \qquad \left(C = -\tfrac{1}{2} f''(\eta) \text{ for some } \eta \text{ between } x^* \text{ and } x_{n-1} \right)$$

$$= K e_{n-1}^2$$

Suppose $K \approx 1$ then if

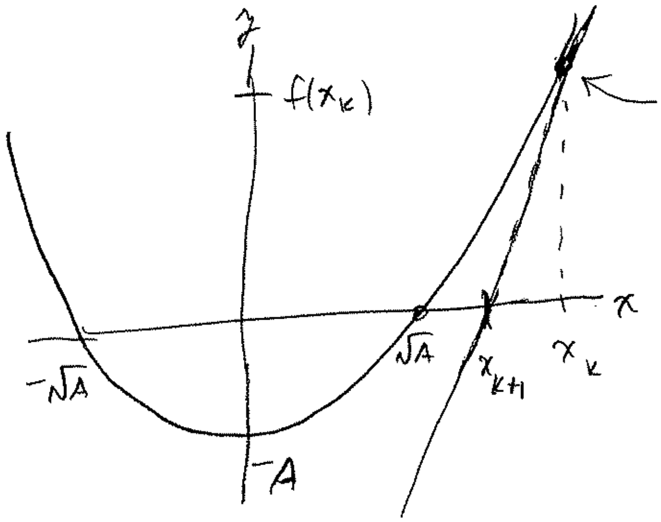
$e_1 \approx 10^{-1}$
$e_2 \approx 10^{-2}$
$e_3 \approx 10^{-4}$
$e_4 \approx 10^{-8}$

quadratic convergence — very fast

i.e. $\boxed{e_n = O(e_{n-1}^2)}$

example Newton's method for finding $\sqrt{A}$ $(A>0)$.

Solve $f(x)=0$ where $f(x)=x^2-A$



tangent line at $(x_k, f(x_k))$

$$y = f(x_k) + f'(x_k)(x-x_k)$$

intersection of x-axis at x_{k+1}

$$0 = f(x_k) + f'(x_k)(x_{k+1}-x_k)$$

$$\Downarrow$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k^2 - A}{2x_k}$$

$$= \frac{1}{2}\left(x_k + \frac{A}{x_k}\right)$$

(Note: $f'(x)$ must exist.)

Take e.g. $A=2$, since $f(0)=-2$ and $f(2)=2$ and f is continuous, by the Intermediate Value Theorem f must have a root in $[0,2]$

We just need an initial guess, e.g. $x_0 = 2$, to get started

simple MATLAB example

$\downarrow$

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  To get started, type one of these: helpwin, helpdesk, or demo.
  For product information, type tour or visit www.mathworks.com.

» format long
» x=2;
» x=(x + 2/x)/2

x =

   1.50000000000000

» x=(x + 2/x)/2

x =

   1.41666666666667

» x=(x + 2/x)/2

x =

   1.41421568627451

» x=(x + 2/x)/2

x =

   1.41421356237469

» x=(x + 2/x)/2

x =

   1.41421356237309

» x=(x + 2/x)/2

x =

   1.41421356237309

» sqrt(2)

ans =

   1.41421356237310
```

Simple MATLAB example: Newton's method for finding $f(x) = x^2 - 2 = 0$ $(x = \sqrt{2})$

initial guess (x=2)

1.50000000000000 ← 1 digit correct, error $= O(10^{-1})$

1.41666666666667 ← 3 digits correct, error $= O(10^{-3})$

1.41421568627451 ← 6 digits correct, error $= O(10^{-6})$

1.41421356237469 ← 12 digits correct, error $= O(10^{-12})$

1.41421356237309 ← almost all available digits correct

What can go wrong with Newton's method?

1.) $f'(x_*) = 0$ or $f'(x_0) \approx 0$ for $f(x_*) = 0$.

If $f'(x_*) = 0$ and f is sufficiently differentiable

then $$f(x) = \underset{\overset{\shortparallel}{0}}{f(x_*)} + \underset{\overset{\shortparallel}{0}}{f'(x_*)}(x - x_*) + \frac{f''(x_*)}{2!}(x-x_*)^2 + O(x-x_*)^3$$

i.e. $$f(x) = \frac{f''(x_*)}{2}(x - x_*)^2 + O(x - x_*)^3 \quad \text{for } x \text{ near } x_*.$$

and $$f'(x) = f''(x_*)(x - x_*) + O(x - x_*)^2 \quad \text{" " "}$$

Assuming $f''(x_*) \neq 0$ Newton's method for

$x = x_k$ near x_* gives

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{\frac{f''(x_*)}{2}(x_k - x_*)^2 + O(x_k - x_*)^3}{f''(x_*)(x_k - x_*) + O(x_k - x_*)^2}$$

$$= x_k - \frac{1}{2}(x_k - x_*) + O(x_k - x_*)^2$$

linear convergence with convergence ratio $\frac{1}{2}$

$\therefore$ error at $k+1$st step

$$= e_{k+1} = x_{k+1} - x_* = \frac{1}{2}(x_k - x_*) + O(x_k - x_*)^2$$

$$\approx \frac{1}{2} e_k = \frac{1}{2} \text{ error at } k\text{th step}$$

---

e.g. $f(x) = x^2$, $f(x_*) = 0$ for $x_* = 0$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2}{2x_k} = x_k - \frac{1}{2}x_k = \frac{1}{2}x_k$$

so $x_{k+1} - x_* = \frac{1}{2}(x_k - x_*)$

```
» format long
» x=.1;
» x=(x + .000001/x)/2

x =

   0.05000500000000

» x=(x + .000001/x)/2

x =

   0.02501249900010

» x=(x + .000001/x)/2

x =

   0.01252623950585

» x=(x + .000001/x)/2

x =

   0.00630303596239

» x=(x + .000001/x)/2

x =

   0.00323084483305

» x=(x + .000001/x)/2

x =

   0.00177018069983

» x=(x + .000001/x)/2

x =

   0.00116754738950

» x=(x + .000001/x)/2

x =

   0.00101202183654

» x=(x + .000001/x)/2

x =

   0.00100007140387

» x=(x + .000001/x)/2

x =

   0.00100000000255

» x=(x + .000001/x)/2

x =

   0.00100000000000
```

Newton's method for

$$f(x) = x^2 - 10^{-6} = 0$$

$$\left(\begin{array}{l} f(x_*) = 0 \quad \text{for} \quad x_* = 10^{-3} = .001 \\ \text{and} \quad f'(x_*) = 2x_* = .002 \approx 0 \end{array} \right)$$

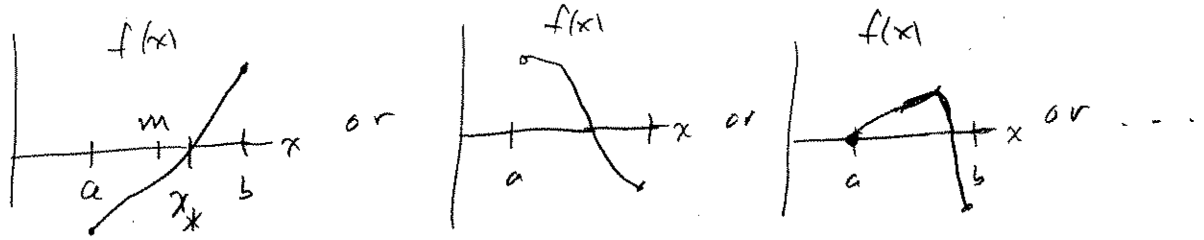
initial convergence linear with factor $\approx \frac{1}{2}$

See Theorem 8 p. 268 in vanLoan's book

final convergence quadratic

Bisection method ~~(see 8.1.2)~~

Suppose $f(x)$ continuous in $[a, b]$ and $f(a)f(b) \le 0$.
Then $f(x) = 0$ for some $x \in [a, b]$.



Let $m = \frac{a+b}{2}$. Then either $f(a)f(m) \le 0$ or $f(m)f(b) \le 0$.
So there is a root in $[a, m]$ or $[m, b]$.

Letting $a_0 = a$, $b_0 = b$,

$$a_{k+1} = a_k, \quad b_{k+1} = \frac{a_k + b_k}{2} \quad \text{if} \quad f(a_k)\, f\left(\frac{a_k + b_k}{2}\right) \le 0$$

and $a_{k+1} = \frac{a_k + b_k}{2}$, $b_{k+1} = b_k$ otherwise

we setting $x_k \equiv (a_k + b_k)/2$ we have

$$|x_{k+1} - x_*| \le \frac{1}{2} |x_k - x_*| \le \frac{1}{2^{k+2}} |a - b|$$

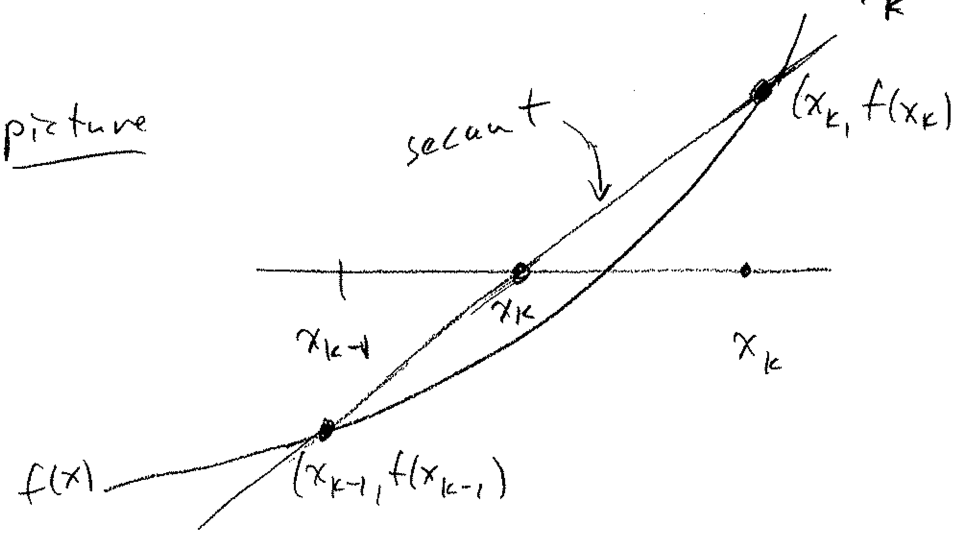
linear convergence with factor $\frac{1}{2}$

~~§1.5~~ Avoiding Derivatives

Secant method $\approx$ Newton's method with

$f'(x_k)$ replaced by $\dfrac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}$

picture



eq. of secant: $y = f(x_k) + \dfrac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}(x - x_k)$

$y=0$ for $x = \boxed{x_{k+1} = x_k - \dfrac{f(x_k)(x_k - x_{k-1})}{f(x_k)-f(x_{k-1})}}$

requires 2 starting values x_0, x_1

and 1 evaluation of $f(x_k)$ per step.

CBS convergence is "superlinear"

$$|x_k - x_*| \leq c\,|x_k - x_*|^r$$

where $r = \dfrac{1+\sqrt{5}}{2} \approx 1.6$ $\therefore$ $1 \leq r \leq 2$

linear conv. quadratic conv.