2/28/08 – 3/4/08 – Math 551 – DeLillo

(1)

PCHIP = Piecewise Cubic Hermite Interpolating Poly.

For $x \in [x_k, x_{k+1}]$, $k = 1, \ldots, n$

$0 \le s = x - x_k \le x_{k+1} - x_k = h_k \; (= h)$

Abusing notation write

$$(P(x) =)\; P(s) = \frac{(3h-2s)s^2}{h^3} y_{k+1} + \frac{h^3 - 3hs^2 + 2s^3}{h^3} y_k$$

$$(*) \qquad + \frac{s^2(s-h)}{h^2} d_{k+1} + \frac{s(s-h)^2}{h^2} d_k$$

Check that $(P(x_k) =)\; P(0) = y_k \qquad P'(0) = d_k$

$(P(x_{k+1}) =)\; P(h) = y_{k+1} \qquad P'(h) = d_{k+1}$

HW due Th 3/6/08

(5) Let $P(s) = c_1 s^3 + c_2 s^2 + c_3 s + c_4$. Using; $P(0) = y_k$, $P'(0) = d_k$, $P(h) = y_{k+1}$; $P'(h) = d_{k+1}$, find c_1, c_2, c_3, c_4 and show that $P(s)$ can be written as above $(*)$.

Note

$$P(s) = \frac{3hs^2 - 2s^3}{h^3} y_{k+1} + \frac{h^3 - 3hs^2 + 2s^3}{h^3} y_k$$

$$+ \frac{s^3 - s^2 h}{h^3} d_{k+1} + \frac{s^3 - 2s^2 h + sh^2}{h^2} d_k$$

$$\therefore P'(s) = \frac{6hs - 6s^2}{h^3} y_{k+1} + \frac{6s^2 - 6hs}{h^3} y_k$$

$$+ \frac{3s^2 - 2sh}{h^2} d_{k+1} + \frac{3s^2 - 4sh + h^2}{h^2} d_k$$

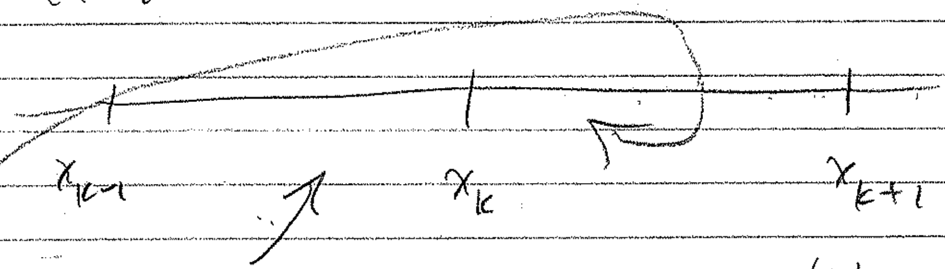
Also for spline we need

$$P''(s) = \frac{6h-12s}{h^3} y_{k+1} + \frac{12s-6h}{h^3} y_k + \frac{6s-2h}{h^2} d_{k+1} + \frac{6s-4h}{h^2} d_k$$

$$= \left(\frac{6h-12s}{h^2}\right)\underbrace{\left(\frac{y_{k+1}-y_k}{h}\right)}_{\delta_k} + \frac{6s-2h}{h^2} d_{k+1} + \frac{6s-4h}{h^2} d_k$$

$$= \frac{(6h-12s)\delta_k + (6s-2h)d_{k+1} + (6s-4h)d_k}{h^2}$$

$(h = h_k)$



$$P''(x_k -) = \frac{(6h_{k-1} - 12h_{k-1})\delta_{k-1} + (6h_{k-1} - 2h_{k-1})d_k}{h_{k-1}^2} + \frac{(6h_{k-1} - 4h_{k-1})d_{k-1}}{h_{k-1}^2}$$

$$= \frac{-6h_{k-1}\delta_{k-1} + 4h_{k-1}d_k + 2h_{k-1}d_{k-1}}{h_{k-1}}$$

$$= \frac{-6\delta_{k-1} + 4d_k + 2d_{k-1}}{h_{k-1}}$$

similarly

$$P''(x_k +) = P''(0) = \frac{6\delta_k - 2d_{k+1} - 4d_k}{h_k}$$

to get $P \in C^2$ (piecewise cubic spline has continuous 2^{nd} derivatives) (3)

We set $P''(x_k -) = P''(x_k +)$, $\underline{k = 2:n-1}$,
then

$$\frac{-6\delta_{k-1} + 4d_k + 2d_{k-1}}{h_{k-1}} = \frac{6\delta_k - 2d_{k+1} - 4d_k}{h_k}$$

or

$$h_k d_{k-1} + 2(h_{k-1} + h_k)d_k + h_{k-1} d_k = 3(h_k \delta_{k-1} + h_{k-1}\delta_k) \quad k = 2, 3, \ldots, n-1$$

Then h_k's and δ_k's are know, so these are $n-2$ linear eqs for n unknown d_k's $k = 1, \ldots, n$

For equally spaced knots x_k we have $h_k = h \; \forall k$ and we set

$$\boxed{d_{k-1} + 4d_k + d_{k+1} = 3(\delta_{k-1} + \delta_k)} \quad (2)$$

We need 2 more eqs coming from conditions near the ends. One choice
(1) is the not-a-knot strategy: Use a single cubic on $x_1 \le x \le x_3$ and $x_{n-2} \le x \le x_n$.

Since $P(x) = P_1(x)$, $x \in [x_1, x_2]$ and $P(x) = P_2(x)$, $x \in [x_2, x_3]$ are cubics with $P_1^{(j)}(x_2) = P_2^{(j)}(x_2)$, $j = 0, 1, 2$, if we set $P_1^{(3)}(x_2) = P'''(x_2 -) = P'''(x_2 +) = P_2^{(3)}(x_2)$

the cubics P_1 and P_2 have to be identical

See HW due 3/6/08 (6.) Prob 3.7 from text

Note $p'''(s) = \dfrac{-12\delta_k + 6d_{k+1} + 6d_k}{h_k^2}$

$\therefore\ p'''(x_2-) = \dfrac{-12\delta_1 + 6d_2 + 6d_1}{h_1^2}$

$= p'''(x_2+) = \dfrac{-12\delta_2 + 6d_3 + 6d_2}{h_1^2}$

If the x_k's are equally spaced $h_k = h$ for all k,

$\therefore\ -2\delta_1 + d_2 + d_1 = -2\delta_2 + d_3 + d_2$

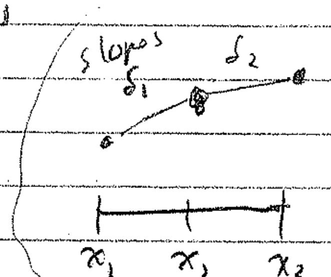
or $\quad d_1 - d_3 = 2\delta_1 - 2\delta_2 \qquad (3)$

Using the condition (2) at $k=2$ above we have

$d_1 + 4d_2 + d_3 = 3(\delta_1 + \delta_2)$

or $\quad d_3 = -d_1 - 4d_2 + 3\delta_1 + 3\delta_2$

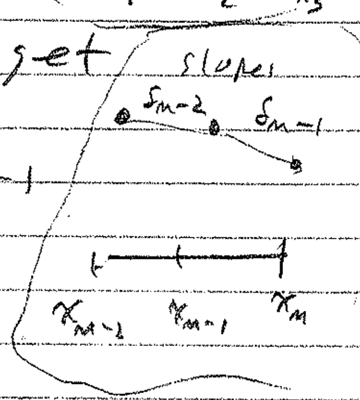
Substituting into (3) gives



$d_1 - d_3 = 2d_1 + 4d_2 = 5\delta_1 + \delta_2$

or $\quad d_1 + 2d_2 = \frac{5}{2}\delta_1 + \frac{1}{2}\delta_2$

similarly at $k = n-1$ we get



$2d_{n-1} + d_n = \frac{1}{2}\delta_{n-2} + \frac{5}{2}\delta_{n-1}$

Our n eqs for $d_1, d_2, \ldots, d_n$ for cubic splines are (with $h_k = h$)

$$d_1 + 2d_2 = \tfrac{5}{2}\delta_1 + \tfrac{1}{2}\delta_2$$

$$d_1 + 4d_2 + d_3 = 3(\delta_1 + \delta_2)$$

$$d_2 + 4d_3 + d_4 = 3(\delta_2 + \delta_3)$$

$$\vdots$$

$$d_{n-2} + 4d_{n-1} + d_n = 3(\delta_{n-2} + \delta_{n-1})$$

$$2d_{n-1} + d_n = \tfrac{1}{2}\delta_{n-2} + \tfrac{5}{2}\delta_{n-1}$$

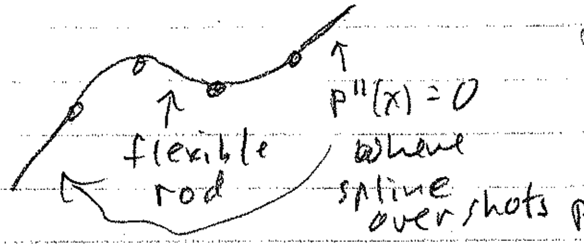
or in matrix-vector form

$$Ad = \begin{bmatrix} 1 & 2 & & & & \\ 1 & 4 & 1 & & 0 & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & 4 & 1 \\ & & & & 2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix} = r = 3\begin{bmatrix} \frac{5}{6}\delta_1 + \frac{1}{6}\delta_2 \\ \delta_1 + \delta_2 \\ \delta_2 + \delta_3 \\ \vdots \\ \delta_{n-2} + \delta_{n-1} \\ \frac{1}{6}\delta_{n-2} + \delta_{n-1} \end{bmatrix}$$

Note, A is tridiagonal

splinetx uses tridisolve to solve for the d_k's.

(2) Another stratedy for choosing the 2 extra equations is the "draftsman's spline":



Use $p''(x_1+) = 0$

$p''(x_n-) = 0$

3. A third choice is Δ-periodicity where

$$\Delta = x_n - x_1 \quad \text{and} \quad P(x+\Delta) = P(x)$$

Now $P(x_1) = y_1 = P(x_n) = y_n$ (restriction on input y_k's)

$P'(x_1) = d_1 = P'(x_n) = d_n$ (eliminates one eq. for d_k's)

The eqs. $d_{k-1} + 4d_k + d_{k+1} = 3(\delta_{k-1} + \delta_k)$

hold for all k now with periodicity at the endpts x_1 and x_n giving

$k=1$

$$d_0 + 4d_1 + d_2 = d_n + 4d_1 + d_2 = 3(\delta_0 + \delta_1) = 3(\delta_{n-1} + \delta_1)$$

and at $k = n-1$

$$d_{n-2} + 4d_{n-1} + d_n = d_{n-2} + 4d_{n-1} + d_1 = 3(\delta_{n-2} + \delta_{n-1})$$

i.e.

$$\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 4 & 1 & & & 0 \\ 0 & 1 & 4 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & 1 & 4 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-2} \\ d_{n-1} \end{bmatrix} = r = 3 \begin{bmatrix} \delta_{n-1} + \delta_1 \\ \delta_1 + \delta_2 \\ \delta_2 + \delta_3 \\ \vdots \\ \delta_{n-2} + \delta_{n-1} \end{bmatrix}$$

$$\underbrace{\qquad}_{Ad}$$

A is an $(n-1) \times (n-1)$ matrix

Solve w $d = A \backslash r$

The case for general $h_k = x_{k+1} - x_k$ with periodic conditions should look like

$$Ad = \begin{bmatrix} 2(h_{m-1}+h_1) & h_{m-1} & & & h_1 \\ h_2 & 2(h_1+h_2) & h_1 & & \\ & h_3 & 2(h_2+h_3) & h_2 & \\ & & \ddots & \ddots & \ddots \\ ? & & & ? & ? \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{m-2} \\ d_{m-1} \end{bmatrix}$$

$(m-1)\times(m-1)$ matrix

You finish this!

$$= r = 3 \begin{bmatrix} h_1\delta_{m-1} + h_{m-1}\delta_1 \\ h_2\delta_1 + h_1\delta_2 \\ h_3\delta_2 + h_2\delta_3 \\ \vdots \end{bmatrix}$$