Fourier Series Methods for Numerical Conformal Mapping of Smooth Domains

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Outline

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   - Numerical preview and gallery
2. Fourier series methods
   - Theodorsen's method (1931)
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   - Multiply connected Fornberg (bounded case, 2009)
3. Remarks and extra details

Outline

Collaborators

Colleagues: Alan Elcrat (WSU) and John Pfaltzgraff (UNC Chapel Hill)

PhD and MS students: Mark Horn, Noureddine Benchama, Lianju (Julian) Wang, and Everett Kropf

General references

[1.] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer, 1964.
[2.] P. Henrici, *Applied and Computational Complex Analysis, Vol. 3*, Wiley, 1986.
[3.] R.Wegmann, *Methods for Numerical Conformal Mapping*, survey article in Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier, 2005, pp. 351–477. Includes presentation of Wegmann's Newton-like methods—similar to ours, but Newton updates are found as solutions to linear Riemann-Hilbert problems on circle domains.

Conformal map $w = f(z)$ from disk to target domain

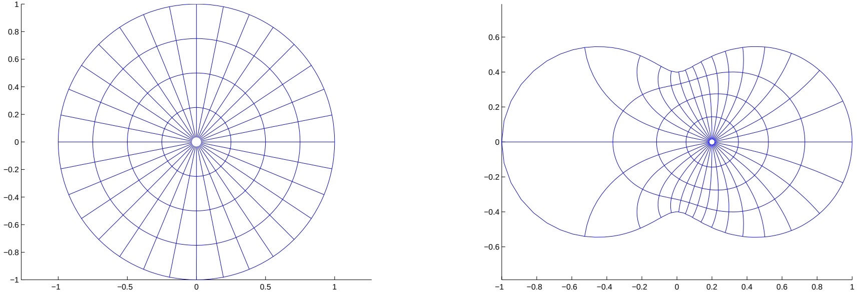


Figure: Fornberg (Fourier series) map from unit disk to interior of an inverted ellipse using 64 Fourier points. $f'(z) \neq 0$, so locally $f(a+h) \approx f(a) + f'(a)h$ and f maps a small circle near $z = a$ to a circle near $f(a)$ magnified by $|f'(a)|$ and rotated by $\arg f'(a)$. Therefore curves intersecting at angle θ at a will be mapped to curves intersecting at angle θ at $f(a)$ and the map is *angle-preserving* or *conformal*. Existence and uniquesness given by Riemann Mapping Theorem with $f(0)$ and $f(1)$ fixed.

Boundary correspondence

The boundary Γ of Ω is parametrized by S (e.g., arclength or polar angle), $\Gamma : \gamma(S), 0 \leq S \leq L, \gamma(0) = \gamma(L)$. If $S = S(\theta)$ or its inverse $\theta(S) = \arg f^{-1}(\gamma(S))$ is known, then the map is known for $z \in D$ or $w \in \Omega$ by the Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma(S(\theta))}{\zeta - z} d\zeta(\theta)$$

or

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{i\theta(S)}}{\gamma(S) - w} d\gamma(S).$$

Two classes of methods

1. Find $S = S(\theta)$ such that $f(e^{i\theta}) = \gamma(S(\theta))$. We will discuss this case. These methods solve a nonlinear integral equation for $S(\theta)$ by linearly convergent methods of successive approximation (Picard-like iteration) such as Theodorsen's method, or quadratically convergent Newton-like methods such as Fornberg's or Wegmann's methods. Cost: $O(N \log N)$ with FFTs.
2. Find $\theta = \theta(S)$ such that $f^{-1}(\gamma(S)) = e^{i\theta(S)}$. These methods solve linear integral equations arising from potential theory for $\theta(S)$ or $\theta'(S)$. Cost: $O(N^2)$ operation counts, but can handle more highly distorted regions.

Two methods for solving nonlinear equations $F(X) = 0$

1. Successive approximation (Picard), if $F(X) = X - G(X)$,

$$X_{n+1} = G(X_n), \quad X_{n+1} \to X_{soln}, \quad \text{converges if} |G'(X_{soln})| < 1.$$

   Less work per step, but convergence is linear.

2. Newton's method, solves linear equation at each step

$$X_{n+1} = X_n - F'(X_n)^{-1} F(X_n).$$

   More work per step, but convergence is quadratic.

Taylor series = Fourier series

For $|z| < |\zeta| = 1, \zeta = e^{i\theta}, d\zeta = ie^{i\theta}d\theta$

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\gamma(S(\theta))}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=1}\gamma(S(\theta))\left(1 + \frac{z}{\zeta} + \left(\frac{z}{\zeta}\right)^2 + \cdots\right)\frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\gamma(S(\theta))(1 + ze^{-i\theta} + z^2e^{-2i\theta} + \cdots)d\theta \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\gamma(S(\theta))e^{-ik\theta}d\theta\right)z^k \\
&= \sum_{k=0}^{\infty}a_k z^k,
\end{aligned}
$$

Taylor coeff. = Fourier coeff. $a_k := \frac{1}{2\pi}\int_0^{2\pi}\gamma(S(\theta))e^{-ik\theta}d\theta.$

Applications:

Transplant boundary value problem for Laplace equation from complicated domain to circle domain or model domain and solve using (fast) Fourier/Laurent series or elementary methods.
(BVP for biharmonic equation can also be solved by transplanting the analytic functions of the Goursat representation.)

Advantages: *fast methods* and *spectral accuracy* for analytic data and boundaries.

Disdavantages: *Crowding phenomenon*–mapping problem can be *severely ill-conditioned* for distorted domains, e.g., an $L \times 1$ elongated domain has derivatives of order $exp(cL)$.

Invariance of Laplacian under $w = f(z)$, conformal

$$\Delta_z U = |f'(z)|^2 \Delta_w U,$$

Therefore, since $f'(z) \neq 0$, $\Delta_z U = 0$ iff $\Delta_w U = 0$.
(Note that for the *biharmonic equation*, $\Delta_w^2 U = 0$, we have

$$\Delta_w^2 U = |f'(z)|^{-2} \Delta_z \left(|f'(z)|^{-2} \Delta_z U \right) = 0,$$

or

$$\Delta_z \left(|f'(z)|^{-2} \Delta_z U \right) = 0.$$

Therefore, the biharmonic equation does not transplant conformally.
However, $U = U(w)$ biharmonic can be written as

$$U = \mathrm{Re}\{\overline{w}\phi(w) + \xi(w)\} = \mathrm{Re}\{\overline{f(z)}\phi(f(z)) + \xi(f(z))\},$$

where ϕ and ξ are the analytic Goursat functions which transplant analytically.)

Outline

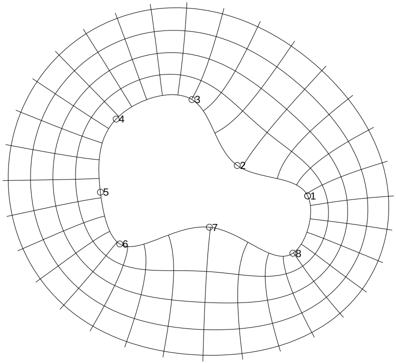


Figure: Fornberg map from exterior of unt disk to exterior of spline

Simply-connected case: crowding=large distortions=Ill-conditioning

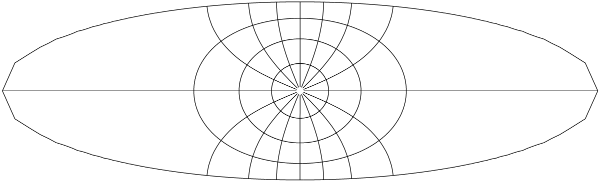


Figure: Fornberg (Fourier series) map from unit disk to interior of ellipse using 1024 Fourier points.

Map from annulus–D. and Pfaltzgraff (1998)

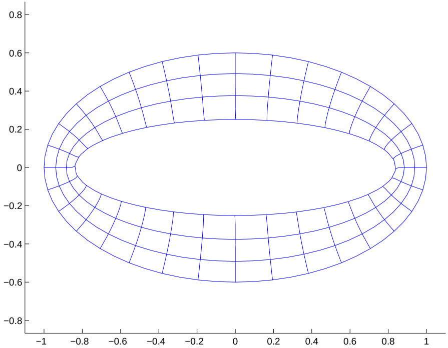


Figure: Doubly connected Fornberg maps annulus $\rho < |z| < 1$ to domain between two ellipses $\alpha = .3, .6$ with $N = 64$. Normalization fixes one boundary point $f(1)$ to fix rotation of annulus. The inner and outer boundary correspondences $S = S_1(\theta)$ and $S = S_2(\theta)$ along with the unique ρ(=1/conformal modulus) must be computed numerically.

Interior mult. conn. case–Kropf's MS thesis (2009)

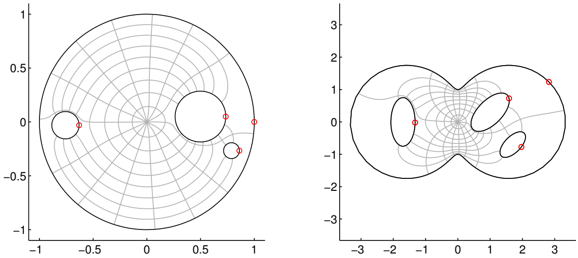
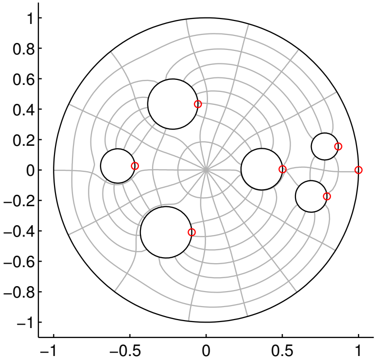
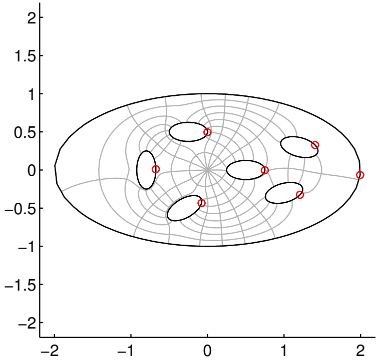


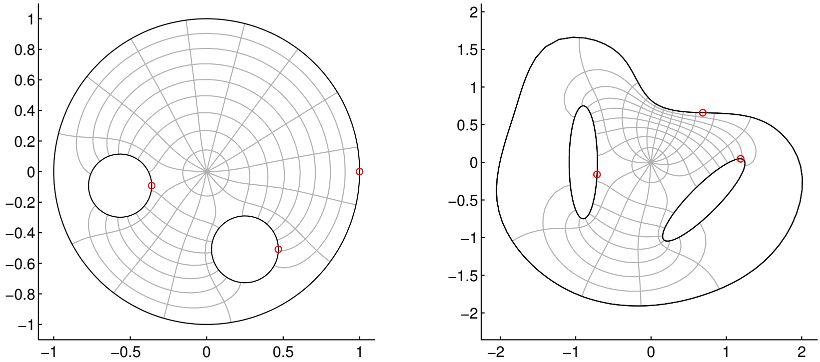
Figure: Outer circle is unit circle. Map normalization fixes $f(0)$ and $f(1)$. $m = 4$ boundary correspondences and centers and radii of inner circles (unique "conformal moduli") must be computed.





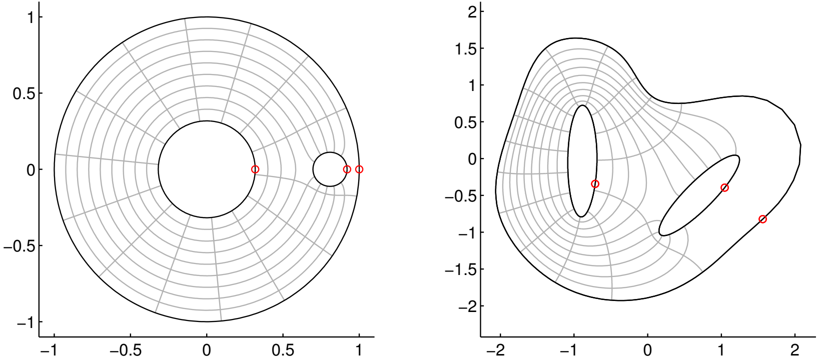
- A target region with $m = 7$.

Numerical Example



- A target region (on the right) with an outer spline boundary which is parametrized by arclength.

Numerical Example



- Annulus with circular holes as a computational domain.

Exterior mult. conn. case–Benchama's PhD thesis (2003)

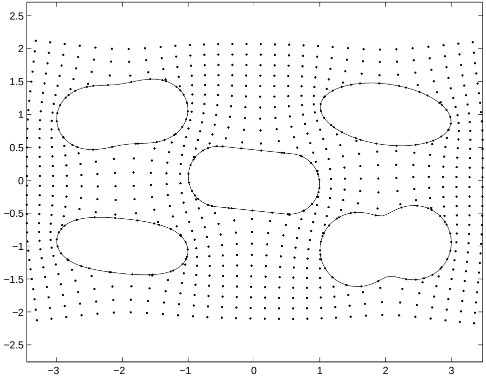


Figure: Fornberg map to the exterior of five curves.

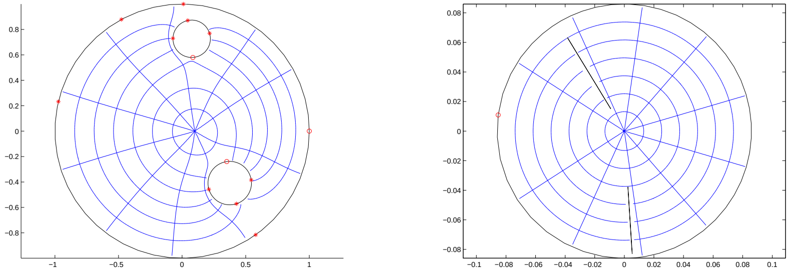


Figure: Infinite product map from circle domain to radial slit disk.

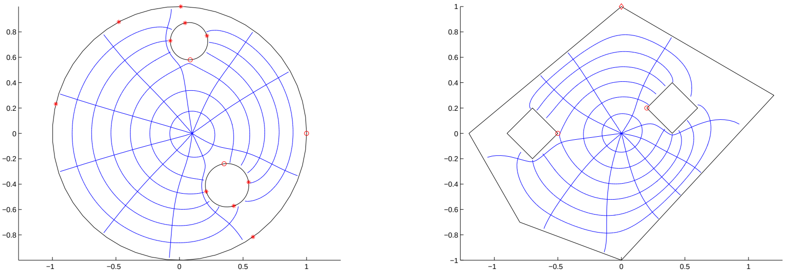


Figure: An orthogonal grid using level lines of map to radial slit disk.

Outline

Conjugate harmonic functions on the disk

Cauchy-Riemann equations in polar coordinates

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

For $u(r,\theta) = r^n \cos(n\theta), n = \ldots, -2, -1, 0, 1, 2, \ldots,$
the *harmonic function conjugate to u* in the disk is

$$v(r,\theta) = r^n \sin(n\theta) + c, \quad c \quad \text{constant.}$$

This gives

$$\begin{aligned} u + iv &= r^n(\cos(n\theta) + i\sin(n\theta)) + ic \\ &= r^n e^{in\theta} + ic = (re^{i\theta})^n + ic = z^n + ic = f(z), \end{aligned}$$

analytic in $z = re^{i\theta}$.
Similarly, if $u(r,\theta) = r^n \sin(n\theta)$, then $v(r,\theta) = -r^n \cos(n\theta) + c$.

Solution of Dirichlet problem on disk

Find $u = u(r,\theta)$ s.t. $\Delta u = 0$ for $0 \le r \le 1$ given (Fourier series for) real boundary data, h,

$$u(1,\theta) = h(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta.$$

The solution is immediate,

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta.$$

For the Dirichlet problem in Ω, we are given boundary values $u = b(S)$ on Γ and transplant to disk, $u(1,\theta) = h(\theta) = b(S(\theta))$.

Computing the conjugate periodic functions

Define the *conjugation operator* K relating conjugate periodic functions, $\phi(\theta) = u(1,\theta)$ and $\psi(\theta) = v(1,\theta) - b_0$,

$$\begin{aligned} \phi(\theta) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta \rightarrow \\ \psi(\theta) &= K\phi(\theta) := \sum_{n=1}^{\infty} a_n \sin n\theta - b_n \cos n\theta. \end{aligned}$$

Therefore, K factors as $\quad K = F^{-1}\hat{K}F,$
where F and F^{-1} are the Fourier transform and it's inverse and

$$\hat{K} = \begin{cases} a_n \rightarrow -b_n \\ a_0 \rightarrow 0 \\ b_n \rightarrow a_n. \end{cases}$$

MATLAB code for conjugation

Note: for complex $h(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, since K is linear,
$Kh(\theta) = \sum_{n=-\infty}^{-1} ia_n e^{in\theta} + \sum_{n=1}^{\infty} -ia_n e^{in\theta}$.

Discretize with N-point trig. interp. and use fft

```
function Kh = conjug(h) % periodic h sampled at n equidistant pts.
n = length(h);
n1 = n/2;
a = fft(h);
a(1) = 0;
a(n1 + 1) = 0;
k = 2:n1;
a(k) = - i*a(k);
a(n1 + k) = i*a(n1 + k);
Kh = ifft(a);
```

Theodorsen's method

Requires that the boundary Γ be *starlike with respect to the origin*, i.e.,

$$\Gamma : \gamma(\phi) = \rho(\phi)e^{i\phi}, 0 < \rho(\phi), 0 \leq \phi \leq 2\pi.$$

The method finds the *boundary correspondence* $\phi = \phi(\theta)$ by successive conjugation.
Start with *auxiliary function* $h(z) := \log f(z)/z$.
Use map normalization $f(0) = 0$ and $f'(0) > 0$.
Note that $h(0) = \log f'(0)$ is real and $h(z)$ is analytic in $|z| < 1$.
Next, note that since $f(e^{i\theta}) = \rho(\phi(\theta))e^{i\phi(\theta)}$, we have

$$\begin{aligned} h(e^{i\theta}) = \log \frac{\rho(\phi(\theta))e^{i\phi(\theta)}}{e^{i\theta}} &= \log \rho(\phi(\theta)) + i(\phi(\theta) - \theta) \\ (&= u(1,\theta) + iv(1,\theta) \quad \text{above.}) \end{aligned}$$

Theodorsen iteration

Apply conjugation operator K to the real and imaginary parts of

$$h(e^{i\theta}) = \log \rho(\phi(\theta)) + i(\phi(\theta) - \theta)$$

Since $\mathrm{Im} h(0) = b_0 = 0$, we have *Theordorsen's equation*,

$$\phi(\theta) - \theta = K[\log \rho(\phi(\theta))]. \tag{1}$$

($-K$ for the exterior case.) Fixing $\phi(0)$ with $0 \geq \phi(0) < 2\pi$ for uniqueness, solve the iteration,

$$\begin{aligned} \phi^{(0)}(\theta) &= \theta \quad \text{(initial guess)} \\ \phi^{(n+1)}(\theta) - \theta &= K[\log \rho(\phi^{(n)}(\theta))]. \end{aligned}$$

Under suitable conditions on Γ, $\phi^{(n)}(\theta) \to \phi^{(exact)}(\theta), n \to \infty$.

K as singular integral operator

$$Kh(\theta) = \frac{1}{2\pi} PV \int_0^{2\pi} h(\tau) \cot\left(\frac{\theta - \tau}{2}\right) d\tau,$$

where PV is the *Cauchy Principal Value* of the integral and $h(\theta)$ is 2π-periodic. (Such singular integral operators are not compact, as we will see.) Define $\delta(\theta) := \phi(\theta) - \theta$. Then $\delta(\theta)$ is 2π-periodic, (whereas, $\phi(\theta)$, of course, is not). Therefore, we actually have *Theodorsen's integral equation* for $\delta = \delta(\theta)$,

$$\delta(\theta) = \frac{1}{2\pi} PV \int_0^{2\pi} \log(\rho(\tau + \delta(\tau)) \cot\left(\frac{\theta - \tau}{2}\right) d\tau$$

Note that this is a *nonlinear* integral equation for $\delta(\theta)$ with the nonlinearity entering through the "curve information" $\log(\rho(\tau + \delta(\tau))$, since K itself is a linear operator.

A useful estimate

Lemma

$\|K\|_2 = 1.$

Proof.

$$\begin{aligned} u(\theta) &\sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \\ \|u\|_2^2 &= |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2, \quad \text{and} \\ \|Ku\|_2^2 &= \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2. \end{aligned}$$

Therefore, $\|Ku\|_2 \leq \|u\|_2$ and if $a_0 = 0$, then $\|Ku\|_2 = \|u\|_2$. Therefore $\|K\|_2 = \max_{\|u\|_2=1} \|Ku\|_2 = 1$. □

Convergence of Theodorsen

Theorem

Let $\epsilon := \sup_\phi |\frac{\rho'(\phi)}{\rho(\theta}|$. *If* $\epsilon < 1$, *then* $\lim_{n\to\infty} \|\phi(\theta) - \phi^{(n)}(\theta)\|_2 \;=\; 0.$

Proof.

From the Theodorsen iteration, we see that

$$\begin{aligned}
\|\phi(\theta) - \phi^{(n+1)}(\theta)\|_2 &= \|K[\log\rho(\phi(\theta)) - \log\rho(\phi^{(n)}(\theta))]\|_2 \\
&\le \|\log\rho(\phi(\theta)) - \log\rho(\phi^{(n)}(\theta))\|_2 \\
&= \|\int_{\phi^{(n)}(\theta)}^{\phi(\theta)} \frac{\rho'(\varphi)}{\rho(\varphi)} d\varphi\|_2 \\
&\le \epsilon\|\phi(\theta) - \phi^{(n)}(\theta)\|_2.
\end{aligned}$$

□

geometric condition for convergece of Theodorsen

$\epsilon < 1$-condition means angle between radial line and normal to curve $< \pi/4$, i.e., Γ is nearly circular.

MATLAB code for Theodorsen's method

```
function f = theoint(n, region, itmax)
th = 2*pi*[0:n-1]/ n; phi = th; phil = phi;
disp('Iteration no. Error between successive iterates');
f = bdrytheo(region,phi);
for it = 1 : itmax
c = log(abs(f));
c = conjug(c);
phi = real(c) + th;
error=max(abs(phi-phil));
phil=phi;
fprintf('
f = bdrytheo(region, phi);
end
```

Popular test case–the inverted ellipse

The map from the interior of the unit disk to the interior of the ellipse $x^2 + \alpha^2 y^2 = 1$ inverted in the unit circle with minor-to-major axis ratio $0 < \alpha \leq 1$ is

$$w = f(z) = \frac{2\alpha z}{1 + \alpha - (1 - \alpha) z^2}.$$

A starlike wrt 0 parametrization of the boundary is

$$\Gamma : \gamma(\phi) = \rho(\phi) e^{i\phi}, 0 \leq \phi \leq 2\pi \quad \text{where} \quad \rho(\phi) = \sqrt{1 - (1 - \alpha^2)\sin^2\phi}.$$

Note: This map can be derived from the Joukowski map $f(z) = z + 1/z$ which maps exteriors of circles to exteriors of ellipses by normalizing properly and rotating.

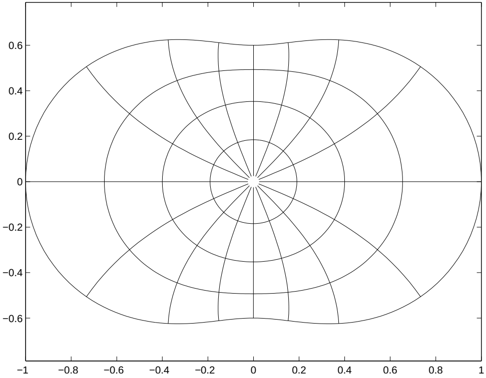


Figure: A target region with an inverted ellipse with $\alpha = .6$. The ϵ-condition is satisfied and Theodorson converged.

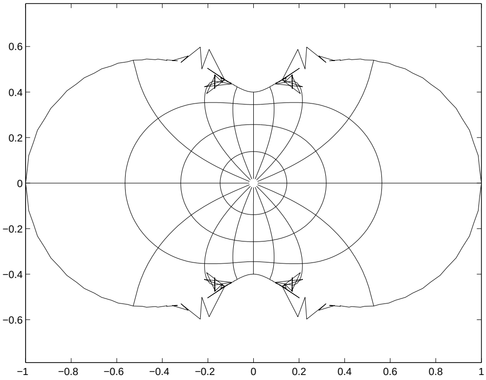


Figure: A target region with an inverted ellipse with $\alpha = .4$. The ϵ-condition is not satisfied and Theodorson failed.

Outline

Conformal map $w = f(z)$ from disk to target domain

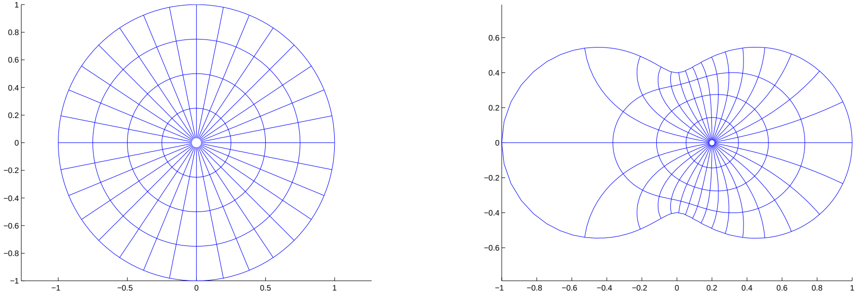


Figure: Fornberg (Fourier series) map from unit disk to interior of an inverted ellipse using 64 Fourier points. Normalization fixes three real parameters: $f(0)$ fixed and $f(1)$ fixed.

Some useful linear operators

For $h = h(\theta), 2\pi$-periodic,

$$
\begin{aligned}
Jh(\theta) &:= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta = c_0 \\
P_+h(\theta) &:= \sum_{k=1}^{\infty} c_k e^{ik\theta} \\
P_-h(\theta) &:= \sum_{k=-\infty}^{0} c_k e^{ik\theta}
\end{aligned}
$$

Note that $P_\pm^2 = P_\pm$ are *projection operators* onto subspaces of $L^2[0, 2\pi]$ whose nonpositive/positive indexed Fourier coefficients 0. Also note

$$
\begin{aligned}
P_+h &= \frac{1}{2}(I + iK - J)h, \\
P_-h &= \frac{1}{2}(I - iK + J)h.
\end{aligned}
$$

(Infinite) matrix form $\quad Fh := \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix} =: \underline{c} \quad$ and $\quad Kh = F^{-1}\hat{K}Fh$

$$= F^{-1} \begin{bmatrix} \ddots & & & & & \\ & i & & & & \\ & & i & & & \\ & & & 0 & & \\ & & & & -i & \\ & & & & & -i \\ & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix} = F^{-1} \begin{bmatrix} \vdots \\ ic_{-2} \\ ic_{-1} \\ 0 \\ -ic_1 \\ -ic_2 \\ \vdots \end{bmatrix},$$

Condition for analytic extension of boundary values

Theorem

A function $h \in Lip(\Gamma)$ *can be continued analytically into* D^+ *(i.e.,* $f(t) = h(t), t \in \Gamma$*) if and only if*

$$f(z) := \frac{1}{2\pi i} \int_\Gamma \frac{h(t)}{t - z} dt = 0, \quad z \in D^-,$$

or, equivalently, if

$$\frac{1}{2\pi i} \int_\Gamma t^n h(t) dt = 0, \quad n = 0, 1, 2, \ldots.$$

Proof.

Sufficiency: Cauchy Integral Theorem.
Necessity: Sokhotskyi jump relations, $f^+ - f^- = h$.

Condition for unit D=disk

Theorem

A function $f \in \mathrm{Lip}(C)$ on the boundary C of the unit disk extends to an analytic function in the interior of the disk with $f(0) = 0$ if and only if

$$P_{-} f(e^{i\theta}) = 0. \tag{2}$$

That is, negative indexed coefficients are 0.

Linearization

Given the kth Newton iterate $S = S^k(\theta)$, find correction $U^k(\theta)$, real, such that

$$f(e^{i\theta}) = \gamma(S^k(\theta) + U^k(\theta)) \approx \xi(\theta) + e^{i\beta(\theta)}U(\theta)$$

extends analytically to the interior of the unit disk with $f(0) = 0$, where $\xi(\theta) = \gamma(S^{(k)}(\theta)), \beta(\theta) = \arg\gamma'(S^{(k)}(\theta))$, and $U(\theta) := |\gamma'(S^{(k)}(\theta)|U^{(k)}(\theta)$ extends analytically to the interior of the unit disk with $f(0) = 0$. The analyticity condition

$$2P_{-}f = (I - iK + J)f = 0$$

implies that

$$(I - iK + J)e^{i\beta(\theta)}U(\theta) = -2P_{-}\xi(\theta).$$

U real gives

$$(I + R)U = r$$

where $R = \mathrm{Re}(e^{-i\beta}(J - iK)e^{i\beta})$ and $r = -\mathrm{Re}(e^{-i\beta}(I - iK + J)\xi)$.

R is a compact operator (Widlund, Wegmann)

$$RU(\theta) := \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin\left(\beta(\phi)-\beta(\theta)+\frac{\theta-\phi}{2}\right)}{\sin\left(\frac{\theta-\phi}{2}\right)} U(\phi)\, d\phi,$$

and for γ sufficiently smooth R^{in} is a symmetric, compact operator on L^2.

Discretization by N-pt. trig. interp.

With $E = \mathrm{diag}_j(e^{i\beta(\theta_j)})$, $j = 0, 1, \cdots, N-1$, discretization gives

$$(I_N + R_N)\underline{U} = \underline{r}.$$

where the matrix

$$I_N + R_N = \frac{2}{N}\mathrm{Re}(E^H F^H P_N F E)$$

(with $P_N := \mathrm{diag}[1, 0, \ldots, 0, 1, \ldots, 1]$) is symmetric and pos.(semi)def. with eigenvalues well-grouped around 1 and conjugate gradient converges superlinearly.
Matrix-vector multiplications costs $O(N \log N)$ with FFT.
The *Newton update* is given by

$$\underline{S}^{(k+1)} = \underline{S}^{(k)} + \underline{U}^{(k)},$$

with $U_0 = 0$ set to fix a boundary point

More details on the matrix-vector formulation

Here $\theta_k = 2\pi k/N$, $0 \le k \le N-1$, so that

$$\underline{f} = [f_0, \ldots, f_{N-1}]^T \qquad f_j = f(e^{i\theta_j}).$$

For $w = e^{2\pi i/N}$, define the Fourier matrix F by

$$F := [w^{-kj}] \qquad 0 \le k, j \le N-1.$$

For $\hat{a}_k$:= kth discrete Fourier coefficients, their N-periodicity $\hat{a}_{k+N} = \hat{a}_k$ gives

$$\frac{1}{N} F \underline{f} = \underline{a} = [\hat{a}_0, \ldots, \hat{a}_{N/2}, \hat{a}_{-N/2+1}, \ldots, \hat{a}_{-1}]^T.$$

Our discrete analyticity conditions are

$$\hat{a}_k = 0, \qquad k = 0, \ldots, -N/2 + 1.$$

Define

$$E = \mathrm{diag}_j[e^{i\beta(\theta_j)}], \qquad 0 \le j \le N-1$$

$$I_1 = \mathrm{diag}[1, 0, \ldots, 0] \qquad I_2 = \mathrm{diag}[0, 1, \ldots, 1]$$

and

$$C = [I_1 \quad I_2] F E$$

where I_1 and I_2 are $N/2 \times N/2$ matrices. Then the inner Newton system is

$$\underline{f} = \underline{\xi} + E\underline{U}$$

and the discrete analyticity conditions are

$$C\underline{U} = -[I_1 \quad I_2] F \underline{\xi} =: \underline{c}.$$

To set $f(1) = \gamma(0)$ requires $S_0 = 0$, and $U_0 = 0$.
Define $\underline{q}^T = [1, 0, ..., 0]$.
Then $U_0 = 0$ is written as $\underline{q}^T \underline{U} = 0$.
Put

$$D = \begin{bmatrix} C \\ \sqrt{N}\underline{q}^T/2 \end{bmatrix}, \quad \underline{g} = \begin{bmatrix} \underline{c} \\ 0 \end{bmatrix}.$$

A calculation gives

$$\frac{2}{N}\mathrm{Re}(D^H D) = \frac{2}{N}\mathrm{Re}(C^H C) + \frac{1}{2}\underline{qq}^T$$

Finally, since $\underline{U}$ is real, we obtain

$$\frac{2}{N}\mathrm{Re}(D^H D)\underline{U} = \frac{2}{N}\mathrm{Re}(D^H \underline{g}). \tag{3}$$

It is useful for visualizing our methods to write the matrices in block form,

$$\begin{aligned} C^H C &= F^H E^H \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} [I_1 \, I_2] FE = F^H E^H \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} FE \\ &= F^H E^H \begin{bmatrix} 1 & & & & & & & \\ & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix} FE. \end{aligned}$$

Outline

Map from annulus–D. and Pfaltzgraff (1998)

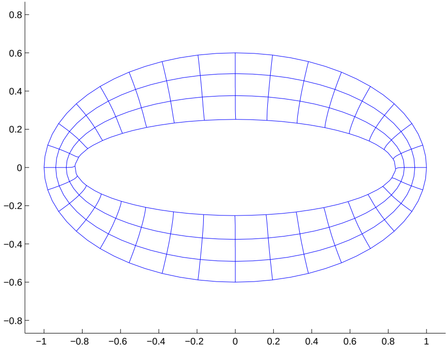


Figure: Doubly connected Fornberg maps annulus $\rho < |z| < 1$ to domain between two ellipses $\alpha = .3, .6$ with $N = 64$. Normalization fixes one boundary point $f(1)$ to fix rotation of annulus. The inner and outer boundary correspondences $S = S_1(\theta)$ and $S = S_2(\theta)$ along with the unique ρ(=1/conformal modulus) must be computed numerically.

Analyticty conditions

Let C_1 and C_2 denote the outer and inner boundaries, respectively, of the annulus $\rho < |z| < 1$, and put $C = C_1 - C_2$.

Theorem

A function $h \in Lip(C)$ extends analytically to the annulus $\rho < |z| < 1$ if and only if

$$\int_{C_1} h(z)z^k dz = \int_{C_2} h(z)z^k dz, \quad k \in \mathbf{Z}.$$

If we let

$$h(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \qquad h(\rho e^{i\theta}) = \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}$$

then the above condition becomes $\rho^k a_k = b_k, \quad k \in \mathbf{Z}$ or (to prevent overflow)

$$\rho^k a_k = b_k, a_{-k} = \rho^k b_{-k}, k = 0, 1, 2, \ldots.$$

Mapping problem

Target region Ω bounded by two smooth curves $\Gamma_1 : \gamma_1(S_1)$ and $\Gamma_2 : \gamma_2(S_2)$.

Problem: Find the *boundary correspondences* $S_1(\theta)$ and $S_2(\theta)$ and the *conformal modulus* ρ such that $f(z)$ is analytic in the annulus $\rho < |z| < 1$ and $f(e^{i\theta}) = \gamma_1(S_1(\theta))$ and $f(\rho e^{i\theta}) = \gamma_2(S_2(\theta))$.

Linearization for Newton-like method

At each Newton step we want to compute corrections $U_1(\theta),\ U_2(\theta)$, and $\delta\rho$ to $S_1(\theta),\ S_2(\theta)$, and ρ. With S_j arclength, $\beta_j(\theta) := \arg\gamma_j'(S_j(\theta)),\ \xi_j(\theta) := \gamma_j(S_j(\theta)),\ j = 1, 2,\ \zeta(\theta) := f'(\rho e^{i\theta})e^{i\theta} = -ie^{i\beta_2(\theta)}dS_2(\theta)/d\theta/\rho$, as in [LM] we linearize about S_1, S_2, and ρ,

$$\begin{aligned} \gamma_j(S_j(\theta) + U_j(\theta)) &\approx \gamma_j(S_j(\theta)) + \gamma_j'(S_j(\theta))U_j(\theta)),\ j = 1, 2, \\ f((\rho + \delta\rho)e^{i\theta}) &\approx f(\rho e^{i\theta}) + f'(\rho e^{i\theta})\delta\rho e^{i\theta} \end{aligned}$$

giving

$$\begin{aligned} f(e^{i\theta}) &\approx \xi_1(\theta) + e^{i\beta_1(\theta)}U_1(\theta) \\ f(\rho e^{i\theta}) &\approx \xi_2(\theta) + e^{i\beta_2(\theta)}U_2(\theta) - \zeta(\theta)\delta\rho. \end{aligned}$$

We find $U_1, U_2, \delta\rho$ to force these BVs to satisfy the analyticity conditions for the annulus.

Discrete form of analyticity conditions

$N-$periodicity of discrete Fourier coefficients $a_{k+N} = a_k$, with $N = 2n$ gives

$$\underline{a} = [a_0, a_1, \ldots, a_n, a_{n+1}, \ldots, a_{N-1}]^T = [a_0, a_1, \ldots, a_n, a_{-n+1}, \ldots, a_{-1}]^T.$$

Define the $N \times N$ matrices $P_1 = \text{diag}[1, \rho, \ldots, \rho^{n-1}, 1, \ldots, 1]$ and $P_2 = -\text{diag}[1, \ldots, 1, 1, \rho^{n-1}, \ldots, \rho]$. Discrete form of our analyticity conditions (with $a_n = b_n$)

$$P_1 \underline{a} + P_2 \underline{b} = 0.$$

Linear equations

With $E_j := \mathrm{diag}_{l=0,\ldots,N-1}[e^{i\beta_j(\theta_l)}],\ j = 1, 2$, our discrete linearizations become

$$N\underline{a} = F\underline{\xi}_1 + FE_1\underline{U}_1$$
$$N\underline{b} = F\underline{\xi}_2 + FE_2\underline{U}_2 - F\underline{\zeta}\delta\rho.$$

Substituting these linearizations into the discrete analyticity conditions gives our linear system for $\underline{U}_1,\ \underline{U}_2$, and $\delta\rho$,

$$[C\ \underline{w}]\underline{U} = P_1FE_1\underline{U}_1 + P_2FE_2\underline{U}_2 - P_2F\underline{\zeta}\delta\rho = -P_1F\underline{\xi}_1 - P_2F\underline{\xi}_2 =: \underline{c}.$$

where $C = [P_1FE_1\ P_2FE_2]$ is a complex $N \times 2N$ matrix, $\underline{w} = -P_2F\underline{\zeta}$ is a complex N-vector, and

$$\underline{U} = \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \\ \delta\rho \end{bmatrix}.$$

This is a system of N complex equations in $2N + 1$ real unknowns, $\underline{U}$.

Normalization

To satisfy the normalization $f(1) = \gamma_1(0)$, we add the equation $\underline{q}^T \underline{U} = U_0 = \delta := 0$, where $\underline{q}^T = [1, 0, \ldots, 0]^T$ is a $2N+1$-vector. We write

$$D = \left[\begin{array}{cc} C & \underline{w} \\ \sqrt{N} & \underline{q}^T/2 \end{array} \right], \; \underline{g} := \left[\begin{array}{c} \underline{c} \\ \delta \end{array} \right].$$

and our system now becomes

$$D\underline{U} = \underline{g},$$

a system of N complex equations and 1 real equation for the $2N+1$ real unknowns, $\underline{U}$. Using the “normal equations” and $\underline{U}$ real, we have

$$\frac{2}{N}\mathrm{Re}(D^H D)\underline{U} = \underline{r} := \frac{2}{N}\mathrm{Re}(D^H \underline{g}).$$

We solve this CG using FFTs.

System = identity + compact

The above $2N+1 \times 2N+1$-matrix is

$$\frac{2}{N}\mathrm{Re}(D^H D) = \begin{bmatrix} A_{11} & A_{12} & \underline{w}_1 \\ A_{12}^T & A_{22} & \underline{w}_2 \\ \underline{w}_1^H & \underline{w}_2^H & 2\underline{w}^H\underline{w}/N \end{bmatrix} + \frac{1}{2}\underline{qq}^T$$

where $A_{ij} = \frac{2}{N}\mathrm{Re}(E_i^H F^H P_i P_j F E_j)$ and $\underline{w}_i = \frac{2}{N}\mathrm{Re}(E_i^H F^H P_i \underline{w}),\ i,j = 1,2.$
Note that the $2N \times 2N$ matrix containing the analyticity conditions is

$$\frac{2}{N}\mathrm{Re}(C^H C) = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}.$$

We'll see $A_{ii} = I+$compact, A_{ij}=compact, $i \neq j$ and $\underline{w}_j$'s are low rank.

Recall $C = [P_1 F E_1 \ P_2 F E_2]$. Then, since $P_i^T = P_i$,

$$C^H C = \begin{bmatrix} E_1^H F^H & 0 \\ 0 & E_2^H F^H \end{bmatrix} \begin{bmatrix} P_1^2 & P_1 P_2 \\ P_2 P_1 & P_2^2 \end{bmatrix} \begin{bmatrix} F E_1 & 0 \\ 0 & F E_2 \end{bmatrix}$$

$P_1^2 = \mathrm{diag}[1, \rho^2, \ldots, \rho^{2(n-1)}, 1, \ldots, 1]$,
$P_2^2 = \mathrm{diag}[1, \ldots, 1, 1, \rho^{2(n-1)}, \ldots, \rho^2]$,
$P_1 P_2 = \mathrm{diag}[1, \rho, \ldots, \rho^{n-1}, 1, \rho^{n-1}, \ldots, \rho]$
The “1” ’s on the diagonals lead to $I + R$ (R compact) as in the disk case.
The ρ^k’s on the diagonals lead to convolutions with, e.g., $l(\theta) = \rho^2 e^{i\theta}/(1 - \rho^2 e^{i\theta}) = \sum_{k=1}^{\infty} \rho^{2k} e^{ik\theta}$.
Therefore, the underlying operator is $I + Compact$, the eigenvalues cluster around 1, and CG converges superlinearly.

Newton update

$$\underline{S}_1^{(k+1)} = \underline{S}_1^{(k)} + \underline{U}_1^{(k)}$$
$$\underline{S}_2^{(k+1)} = \underline{S}_2^{(k)} + \underline{U}_2^{(k)}$$
$$\rho^{(k+1)} = \rho^{(k)} + \delta\rho^{(k)}.$$

Outline

Interior mult. conn. case–Kropf's MS thesis (2009)

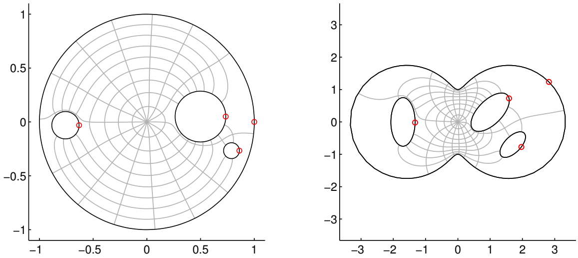
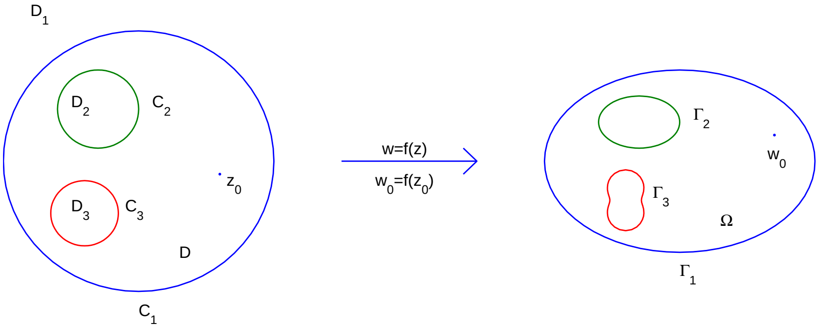


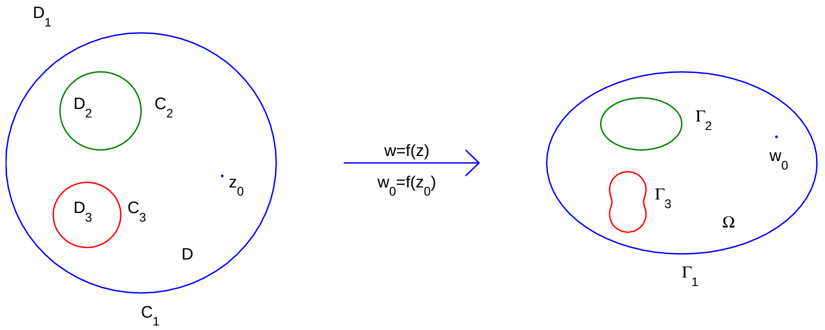
Figure: Outer circle is unit circle. Map normalization fixes $f(0)$ and $f(1)$. $m = 4$ boundary correspondences and centers and radii of inner circles (unique "conformal moduli") must be computed.

Computational and Target Domains



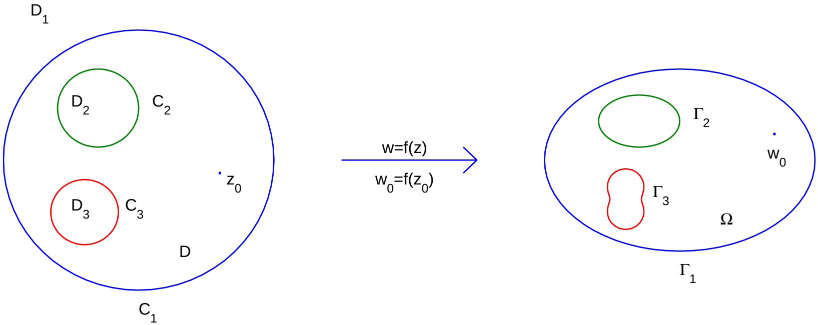
- The boundary of the computational domain D is $C = C_1 - \cdots - C_m$,
  - where m is the connectivity of D
  - and C_1 is the unit circle.
- The boundary of the target ("physical") domain Ω is $\Gamma = \Gamma_1 - \cdots - \Gamma_m$.

Boundary Parametrization



- The target domain boundary will be parametrized, e.g., by arclength,
- *i.e.*, $\Gamma : \gamma_1(S_1) - \gamma_2(S_2) - \cdots - \gamma_m(S_m)$.

Computational Goal



- The goal is to compute the conformal map $f : D \to \Omega$.
- To do this we must calculate
  1. the centers c_ν and radii ρ_ν of the circles C_ν, $2 \le \nu \le m$, and
  2. the boundary correspondences $S_\nu(\theta)$, where $0 \le \theta \le 2\pi$,

such that $f(c_\nu + \rho_\nu e^{i\theta}) = \gamma_\nu(S_\nu(\theta))$, $1 \le \nu \le m$.

A Newton-like Method

The desired map will be computed using a Newton-like method:

1. Begin with an initial guess for the centers c_ν and radii ρ_ν, and the boundary correspondences $S_\nu(\theta)$.
2. Using linearized version of the circle map problem, find updates to these values by solving a linear system.
3. Apply the updates.
4. Keep doing this until the updates found are below some specified value.
5. Based on the result of the last Newton iteration, calculate the the map.

Form of the Map

Theorem

The conformal map described above has the series representation

$$f(z) = \sum_{j=0}^{\infty} a_{1,j} z^j + \sum_{\nu=2}^{m} \sum_{j=1}^{\infty} a_{\nu,-j} \left(\frac{\rho_\nu}{z - c_\nu} \right)^j,$$

where for $1 \leq \nu \leq m$ *and* $j > 0$ *the Fourier coefficients* $a_{\nu,j}$ *are defined*

$$a_{\nu,j} := \frac{1}{2\pi} \int_0^{2\pi} f(c_\nu + \rho_\nu e^{i\theta}) e^{-ij\theta} \, d\theta.$$

Proof of the Form of the Map

(part 1)

Proof.

For a point z in D (with z not on the boundary) the Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \sum_{\nu=2}^{m}\frac{1}{2\pi i}\int_{C_\nu}\frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof of the Form of the Map

(part 2)

Proof.

Note that $\zeta = e^{i\theta} \Rightarrow d\zeta = ie^{i\theta}\, d\theta$, along with $\frac{|z|}{|\zeta|} < 1$. Expanding the Cauchy kernel around C_1 gives

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{C_1} f(\zeta)\frac{1}{1 - z/\zeta}\frac{d\zeta}{\zeta}$$

$$= \frac{1}{2\pi i}\int_{C_1} f(\zeta)\sum_{j=0}^{\infty}\left(\frac{z}{\zeta}\right)^j \frac{d\zeta}{\zeta}$$

$$= \sum_{j=0}^{\infty}\left[z^j \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ij\theta}\, d\theta\right].$$

Proof of the Form of the Map

(part 3)

Proof.

Additionally $\zeta = c_\nu + \rho_\nu e^{i\theta} \Rightarrow d\zeta = i\rho_\nu e^{i\theta}\, d\theta$, and $\frac{|\zeta - c_\nu|}{|z - c_\nu|} < 1$. So on each C_ν

$$\frac{1}{2\pi i}\int_{C_\nu} \frac{f(\zeta)}{\zeta - c_\nu - (z - c_\nu)}\, d\zeta = -\frac{1}{2\pi i}\int_{C_\nu} f(\zeta)\frac{1}{z - c_\nu}\sum_{j=0}^{\infty}\left(\frac{\zeta - c_\nu}{z - c_\nu}\right)^j d\zeta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} f(c_\nu + \rho_\nu e^{i\theta})\frac{1}{z - c_\nu}\sum_{j=0}^{\infty}\left(\frac{\rho_\nu e^{i\theta}}{z - c_\nu}\right)^j \rho_\nu e^{i\theta}\, d\theta$$

$$= -\sum_{j=1}^{\infty}\left[\left(\frac{\rho_\nu}{z - c_\nu}\right)^j \frac{1}{2\pi}\int_0^{2\pi} f(c_\nu + \rho_\nu e^{i\theta}) e^{ij\theta}\, d\theta\right].$$

□

Analytic Continuation

Theorem

Let C be a positively oriented, Lipschitz continuous curve with D the region bounded by C and D^- the compliment of $D \cup C$. A function $f \in \mathrm{Lip}(C)$ can be continued analytically into D if and only if

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0, \quad \forall z \in D^-.$$

- A version of this theorem is given by both Henrici and Muskhelishvili.
- It is used here as setup for the next theorem
- where we introduce the conditions for analytic extention (analyticity conditions).

Analyticity Conditions

Theorem

A function $f \in \mathrm{Lip}(C)$ extends analytically into D if and only if for all $k \geq 0$

$$a_{1,-(k+1)} - \sum_{\nu=2}^{m} \sum_{j=0}^{k} \binom{k}{j} \rho_\nu^{j+1} c_\nu^{k-j} a_{\nu,-(j+1)} = 0$$

and

$$\sum_{j=0}^{\infty} B_{k+1,j} \rho_\ell^k c_\ell^j a_{1,k+j} - a_{\ell,k} - \sum_{\substack{\nu=2 \\ \nu \neq \ell}}^{m} \sum_{j=0}^{\infty} \frac{\rho_\ell^k}{(c_\nu - c_\ell)^{k+1}} B_{k+1,j} \frac{\rho_\nu^{j+1}}{(c_\ell - c_\nu)^j} a_{\nu,-(j+1)} = 0.$$

Note on Analyticity Conditions

For the analyticity conditions we need to define some binomial coefficients.

Definition

For $k > 0$ and $x, y \in \mathbb{C}$,

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j \quad \text{where} \quad \binom{k}{j} := \frac{k!}{j!(k-j)!}.$$

Definition

For $k > 0$ and $|z| < 1$,

$$\frac{1}{(1-z)^k} = \sum_{j=0}^{\infty} B_{k,j} z^j \quad \text{where} \quad B_{k,j} := \frac{k(k+1)\cdots(k+j-1)}{j!}.$$

Note on Proof of Analyticity Conditions

The proof involves

1. applying the above analytic continuation Theorem for an arbitrary point z in each $D_1, \ldots, D_m$,
2. expanding the function in the appropriate Laurent series, and
3. setting the resulting series equal to 0.

Proof of Analyticity Conditions

(Outside C_1)

Proof.

For z in D_1 we have $|z| > 1$ and $|\zeta|/|z| < 1$ for ζ on any $C_1, \ldots, C_m$, thus

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = -\frac{1}{2\pi i}\int_C f(\zeta)\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{\zeta}{z}\right)^k d\zeta$$

$$= -\sum_{k=0}^{\infty} z^{-k-1}\frac{1}{2\pi i}\int_C f(\zeta)\zeta^k\,d\zeta = 0.$$

The last integral on the RHS must be zero for all $k \geq 0$.

Proof of Analyticity Conditions

(Outside C_1)

Proof.

$$0 = \frac{1}{2\pi i}\int_C f(\zeta)\zeta^k\,d\zeta = \frac{1}{2\pi i}\int_{C_1} f(\zeta)\zeta^k\,d\zeta - \sum_{\nu=2}^{m}\frac{1}{2\pi i}\int_{C_\nu} f(\zeta)\zeta^k\,d\zeta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{i(k+1)\theta}\,d\theta$$

$$-\sum_{\nu=2}^{m}\sum_{j=0}^{k}\binom{k}{j}\rho_\nu^{j+1}c_\nu^{k-j}\frac{1}{2\pi}\int_0^{2\pi} f(c_\nu+\rho_\nu e^{i\theta})e^{i(j+1)\theta}\,d\theta$$

$$= a_{1,-(k+1)} - \sum_{\nu=2}^{m}\sum_{j=0}^{k}\binom{k}{j}\rho_\nu^{j+1}c_\nu^{k-j}a_{\nu,-(j+1)}.$$

Proof of Analyticity Conditions

(Inside C_ℓ, $2 \le \ell \le m$)

Proof.

For z in one of D_ℓ we have $|z - c_\ell|/|\zeta - c_\ell| < 1$ for ζ on any $C_1, \ldots, C_m$, and so

$$0 = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - c_\ell - (z - c_\ell)}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_C f(\zeta)\frac{1}{\zeta - c_\ell}\sum_{k=0}^{\infty}\left(\frac{z - c_\ell}{\zeta - c_\ell}\right)^k\,d\zeta$$

$$= \sum_{k=0}^{\infty}(z - c_\ell)^k \frac{1}{2\pi i}\int_C f(\zeta)(\zeta - c_\ell)^{-k-1}\,d\zeta.$$

Again the last integral on the RHS must be zero for all $k \ge 0$.

Proof of Analyticity Conditions

(Inside C_ℓ, $2 \le \ell \le m$)

Proof.

Thus around C_1

$$\frac{1}{2\pi i}\int_{C_1} f(\zeta)(\zeta - c_\ell)^{-k-1}\,d\zeta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})(e^{i\theta} - c_\ell)^{-k-1} e^{i\theta}\,d\theta$$

$$= \sum_{j=0}^{\infty} B_{k+1,j} c_\ell^j \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) e^{-i(k+j)\theta}\,d\theta.$$

Proof of Analyticity Conditions

(Inside C_ℓ, $2 \le \ell \le m$)

Proof.

To expand the previous integral we had to apply the binomial theorem.

When integrating around C_1, $|c_\ell|/|e^{i\theta}| < 1$ and so

$$(e^{i\theta} - c_\ell)^{-k-1} = \frac{1}{e^{i(k+1)\theta}} \cdot \frac{1}{\left(1 - \frac{c_\ell}{e^{i\theta}}\right)^{k+1}} = \frac{1}{e^{i(k+1)\theta}} \sum_{j=0}^{\infty} B_{k+1,j} \left(\frac{c_\ell}{e^{i\theta}}\right)^j .$$

Proof of Analyticity Conditions

(Inside C_ℓ, $2 \leq \ell \leq m$)

Proof.

Around C_ν, $2 \leq (\nu \neq \ell) \leq m$

$$\frac{1}{2\pi i}\int_{C_\nu} f(\zeta)(\zeta - c_\ell)^{-k-1}\, d\zeta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(c_\nu + \rho_\nu e^{i\theta})(\rho_\nu e^{i\theta} - (c_\ell - c_\nu))^{-k-1} \rho_\nu e^{i\theta}\, d\theta$$

$$= \frac{1}{(c_\nu - c_\ell)^{k+1}} \sum_{j=0}^{\infty} B_{k+1,j} \frac{\rho_\nu^{j+1}}{(c_\ell - c_\nu)^j} \frac{1}{2\pi}\int_0^{2\pi} f(c_\nu + \rho_\nu e^{i\theta}) e^{i(j+1)\theta}\, d\theta.$$

Proof of Analyticity Conditions

(Inside C_ℓ, $2 \leq \ell \leq m$)

Proof.

Again the binomial theorem was applied.

Since $\rho_\nu / |c_\ell - c_\nu| < 1$ around C_ν for $2 \leq (\nu \neq \ell) \leq m$, we have

$$(\rho_\nu e^{i\theta} - (c_\ell - c_\nu))^{-k-1} = \frac{1}{(c_\ell - c_\nu)^{k+1}(-1)^{k+1}\left(1 - \frac{\rho_\nu e^{i\theta}}{c_\ell - c_\nu}\right)^{k+1}}$$

$$= \frac{1}{(c_\nu - c_\ell)^{k+1}} \sum_{j=0}^{\infty} B_{k+1,j} \left(\frac{\rho_\nu e^{i\theta}}{c_\ell - c_\nu}\right)^j .$$

Proof of Analyticity Conditions

(Inside C_ℓ, $2 \le \ell \le m$)

Proof.

And finally, around C_ℓ

$$\frac{1}{2\pi i}\int_{C_\ell} f(\zeta)(\zeta - c_\ell)^{-k-1}\, d\zeta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(c_\ell + \rho_\ell e^{i\theta})\rho_\ell^{-k-1} e^{-i(k+1)\theta}\rho_\ell e^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(c_\ell + \rho_\ell e^{i\theta})\rho_\ell^{-k} e^{-ik\theta}\, d\theta.$$

Proof of Analyticity Conditions

(Inside C_ℓ, $2 \le \ell \le m$)

Proof.

Putting it together,

$$0 = \frac{1}{2\pi i}\int_C f(\zeta)(\zeta - c_\ell)^{-k-1}\,d\zeta$$

$$= \sum_{j=0}^{\infty} B_{k+1,j}\rho_\ell^k c_\ell^j a_{1,k+j} - a_{\ell,k}$$

$$- \sum_{\substack{\nu=2\\ \nu\neq\ell}}^{m}\sum_{j=0}^{\infty} \frac{\rho_\ell^k}{(c_\nu - c_\ell)^{k+1}} B_{k+1,j}\frac{\rho_\nu^{j+1}}{(c_\ell - c_\nu)^j} a_{\nu,-(j+1)}.$$

□

Map Normalization

- The map is normalized by specifying three real conditions.
  - One is given by specifying $f(1) = \gamma_1(0)$.
  - The other two are given by fixing $f(z_0) = w_0$ for points $z_0 \in D$ and $w_0 \in \Omega$. This is given by the form of the map previously calculated, *i.e.*

$$w_0 = f(z_0) = \sum_{k=0}^{\infty} a_{1,k} z_0^k + \sum_{\nu=2}^{m} \sum_{k=1}^{\infty} a_{\nu,-k} \left(\frac{\rho_\nu}{z_0 - c_\nu} \right)^k .$$

A Newton-like Method

The desired map will be computed using a Newton-like iteration:

1. Begin with an initial guess for the centers c_ν and radii ρ_ν, and the boundary correspondences $S_\nu(\theta)$.
2. Using a discretized version of the analyticity conditions and normalization conditions, and a linearized version of the circle map problem, find updates to these values by solving a linear system.
3. Apply the updates.
4. Keep doing this until the updates found are below some specified value.
5. Based on the result of the last Newton iteration, calculate the Fourier coefficients to form the map.

Linearization

We now write $f(c_\nu + \rho_\nu e^{i\theta}) = \gamma_\nu(S_\nu(\theta))$ as a linear problem.

- For an initial guess $S_\nu(\theta)$ and 2π periodic correction $U_\nu(\theta)$,

$$\gamma_\nu(S_\nu(\theta) + U_\nu(\theta)) \approx \gamma_\nu(S_\nu(\theta)) + \gamma_\nu'(S_\nu(\theta))U_\nu(\theta).$$

- For an initial guess of c_ν and ρ_ν with corrections δc_ν and $\delta\rho_\nu$,

$$\begin{aligned}(f + \delta f)(c_\nu + \delta c_\nu + (\rho_\nu + \delta\rho_\nu)e^{i\theta})& \\ \approx (f + \delta f)(c_\nu + \rho_\nu e^{i\theta}) &+ f'(c_\nu + \rho_\nu e^{i\theta})(\delta c_\nu + \delta\rho_\nu e^{i\theta}).\end{aligned}$$

- Setting the RHS of these approximations equal gives

$$\begin{aligned}(f + \delta f)(c_\nu + \rho_\nu e^{i\theta}) = \gamma_\nu(S_\nu(\theta)) &+ \gamma_\nu'(S_\nu(\theta))U_\nu(\theta) \\ &- f'(c_\nu + \rho_\nu e^{i\theta})(\delta c_\nu + \delta\rho_\nu e^{i\theta}).\end{aligned}$$

Linearization

More concisely

- For convenience define
  - $\xi_\nu(\theta) := \gamma_\nu(S_\nu(\theta))$,
  - $\eta_\nu(\theta) := \gamma'_\nu(S_\nu(\theta))$, and
  - $\zeta_\nu(\theta) := -f'(c_\nu + \rho_\nu e^{i\theta})e^{i\theta} = i\rho_\nu^{-1}\eta_\nu S'_\nu(\theta)$.
- The linearization conditions can then be written
  - $(f + \delta f)(e^{i\theta}) = \xi_1(\theta) + \eta_1(\theta)U_1(\theta)$
  - $(f + \delta f)(c_\nu + \rho_\nu e^{i\theta}) = \xi_\nu(\theta) + \eta_\nu(\theta)U_\nu(\theta) + \zeta_\nu(\theta)(\delta\rho_\nu + \delta c_\nu e^{-i\theta})$

  for the updates around C_1 and around C_ν, $2 \le \nu \le m$, respectively.

Newton Updates

- After the linear system has been solved, the updates are applied at each step (n) as follows:
  - $S_\nu^{(n)}(\theta) = S_\nu^{(n-1)}(\theta) + U_\nu^{(n-1)}(\theta)$

  for $1 \leq \nu \leq m$, and
  - $c_\nu^{(n)} = c_\nu^{(n-1)} + \delta c_\nu^{(n-1)}$
  - $\rho_\nu^{(n)} = \rho_\nu^{(n-1)} + \delta \rho_\nu^{(n-1)}$

  for $2 \leq \nu \leq m$.

N Discrete Fourier Coefficients

- Let N be an even number.
- Let $a_{1,k}, \ldots, a_{m,k}$ now denote the discrete Fourier coefficients.
- The N-periodicity of the discrete coefficients, with $M = N/2$, gives

$$\underline{a}_\nu := (a_{\nu,0}, a_{\nu,1}, \ldots, a_{\nu,N-1})^T$$
$$= (a_{\nu,0}, \ldots, a_{\nu,M-1}, a_{\nu,-M}, \ldots, a_{\nu,-1})^T$$

for $1 \leq \nu \leq m$.

N-point Discretization

- Again with $M = N/2$ we discretize the analyticity and normalization conditions by
  - limiting both the analyticity and normalization conditions to M terms in each sum expansion, and
  - limiting the analyticity conditions to M equations.
- This can be done by making $k = 0, \ldots, M-1$ or $k = 1, \ldots, M$ as appropriate. The result is the discrete system of equations ...

Discrete System of Equations

- $$a_{1,-(k+1)} - \sum_{\nu=2}^{m} \sum_{j=0}^{k} \binom{k}{j} \rho_\nu^{j+1} c_\nu^{k-j} a_{\nu,-(j+1)} = 0,$$

- $$\sum_{j=0}^{M-1} B_{k+1,j} \rho_\ell^k c_\ell^j a_{1,k+j} - a_{\ell,k}$$
$$- \sum_{\substack{\nu=2 \\ \nu\neq\ell}}^{m} \sum_{j=0}^{M-1} \frac{\rho_\ell^k}{(c_\nu - c_\ell)^{k+1}} B_{k+1,j} \frac{\rho_\nu^{j+1}}{(c_\ell - c_\nu)^j} a_{\nu,-(j+1)} = 0,$$

- $$\sum_{j=0}^{M-1} a_{1,j} z_0^j + \sum_{\nu=2}^{m} \sum_{j=1}^{M} a_{\nu,-j} \left(\frac{\rho_\nu}{z_0 - c_\nu} \right)^j = w_0.$$

Matrix Form

of the Analyticity and Normalization Conditions

- The discrete system of equations can be written

$$P\underline{a} = P_1\underline{a}_1 + \cdots + P_m\underline{a}_m = [P_1 \quad \cdots \quad P_m] \begin{bmatrix} \underline{a}_1 \\ \vdots \\ \underline{a}_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ w_0 \end{bmatrix} := \underline{r}.$$

Discrete Linearization Conditions

- We need to define the vectors and vector functions
  - $\underline{\theta} := \frac{2\pi}{N}(0, 1, \ldots, N-1)^T$,
  - $\underline{\xi}_\nu := \xi_\nu(\underline{\theta})$,
  - and similarly for $\underline{\eta}_\nu$, $\underline{\zeta}_\nu$, and $\underline{U}_\nu$.
- If F is the discrete Fourier transform matrix, $E_\nu := \mathrm{diag}(\underline{\eta}_\nu)$, $\underline{q} := e^{-i\underline{\theta}}$, and $*$ is the Hadamard product, then the linearization conditions become
  - $N\underline{a}_1 = F\underline{\xi}_1 + FE_1\underline{U}_1$ and
  - $N\underline{a}_\nu = F\underline{\xi}_\nu + FE_\nu\underline{U}_\nu + \delta\rho_\nu F\underline{\zeta}_\nu + \delta c_\nu F(\underline{q} * \underline{\zeta}_\nu)$.

New Linear System

- For ease of exposition, assume $m = 3$ for the rest of this section.
- Combining the discrete system of equations for the analyticity and normalization conditions with the discretized linear conditions gives

$$
\begin{aligned}
&P_1 F E_1 \underline{U}_1 \\
&\quad + P_2(F E_2 \underline{U}_2 + \delta\rho_2 F \underline{\zeta}_2 + (\mathrm{Re}\,\delta c_2 + i\mathrm{Im}\,\delta c_2) F(\underline{q} * \underline{\zeta}_2)) \\
&\quad + P_3(F E_2 \underline{U}_3 + \delta\rho_3 F \underline{\zeta}_3 + (\mathrm{Re}\,\delta c_3 + i\mathrm{Im}\,\delta c_3) F(\underline{q} * \underline{\zeta}_3)) \\
&\qquad = N\underline{r} - P_1 F \underline{\xi}_1 - P_2 F \underline{\xi}_2 - P_3 F \underline{\xi}_3 := \underline{\tilde{g}}.
\end{aligned}
$$

More Convenience Notation

- Let $\underline{w}_\nu := P_\nu F \underline{\zeta}_\nu$,
- $\underline{wq}_\nu := P_\nu F(\underline{q} * \underline{\zeta}_\nu)$,
- $W := \begin{bmatrix} \underline{w}_2 & \underline{w}_3 & \underline{wq}_2 & i\underline{wq}_2 & \underline{wq}_3 & i\underline{wq}_3 \end{bmatrix}$,
- and of course $P := \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$.
- Also define the real vector $\underline{U} :=$

$$\begin{bmatrix} \underline{U}_1^T & \underline{U}_2^T & \underline{U}_3^T & \delta\rho_2 & \delta\rho_3 & \mathrm{Re}\,\delta c_2 & \mathrm{Im}\,\delta c_2 & \mathrm{Re}\,\delta c_3 & \mathrm{Im}\,\delta c_3 \end{bmatrix}^T.$$

The Matrix $\tilde{D}$

- Combining all of this we now have

$$\tilde{D}\underline{U} := \begin{bmatrix} P_1 & P_2 & P_3 & W \end{bmatrix} \begin{bmatrix} F & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \underline{U} = \underline{\tilde{g}}.$$

The Matrix D

Through normalization

- We add a row to this system to force $U_1(0) = 0$ at every iteration.
- This satisfies the normalization condition $f(1) = \gamma_1(0)$.
- To do this define the vector $\underline{v}^T := (1, 0, \ldots, 0)$, and then

$$D := \begin{bmatrix} \tilde{D} \\ \frac{\sqrt{N}}{2}\underline{v}^T \end{bmatrix} \quad \text{and} \quad \underline{g} := \begin{bmatrix} \tilde{g} \\ 0 \end{bmatrix}.$$

The Matrix A

- Taking the "normal equations" and using the fact $\underline{U}$ is real,

$$A\underline{U} := \frac{2}{N}\mathrm{Re}\,(D^H D)\underline{U} = \frac{2}{N}\mathrm{Re}\,(D^H \underline{g}) := \underline{b}.$$

- This system can now be solved efficiently using the conjugate gradient method.

The Matrix A Decomposed

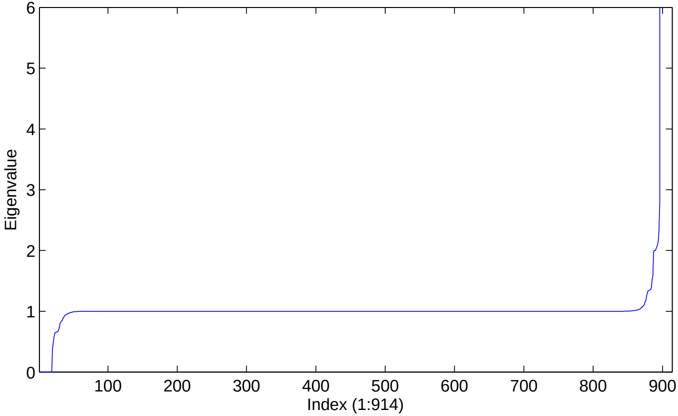
- Define
  - $A_{kj} := (2/N)\mathrm{Re}\,(E_k^H F^H P_k^H P_j F E_j)$ and
  - $X_k := (2/N)\mathrm{Re}\,(E_k^H F^H P_k^H W)$.
- Then A can be written

$$A = \frac{2}{N}\mathrm{Re}\,(D^H D) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & X_1 \\ A_{21} & A_{22} & A_{23} & X_2 \\ A_{31} & A_{32} & A_{33} & X_3 \\ X_1^T & X_2^T & X_3^T & W^H W \end{bmatrix} + \frac{1}{2}\underline{v v^T},$$

Eigenvalues of A

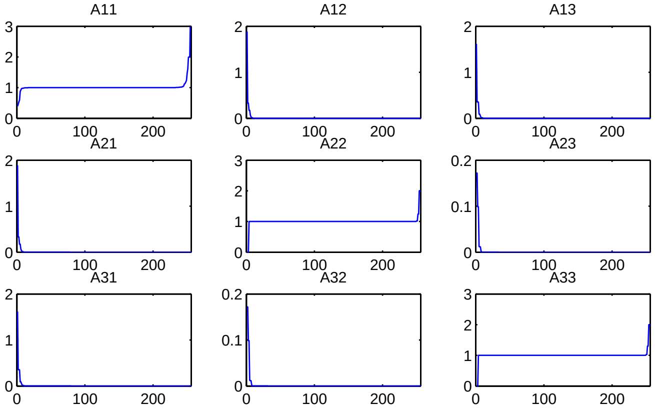
- To understand the eigenvalues of A it suffices to examine the submatrix
$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$
- For the eigenvalues:
  - The diagonal entries can be shown to be discretizations of the identity plus a compact operator, and
  - the off-diagonal entries can be shown to be discretizations of a compact operator.
- In effect $\hat{A}$ is a low-rank perturbation of the identity, and the eigenvalues cluster around 1.
- This is the property which makes the conjugate gradient method an efficient solver to use for this problem.

Eigenvalues of A Cluster Around 1



- This map had $m = 7$ and $N = 128$.

Eigenvalues of $\hat{A}$



- This map had connectivity $m = 3$ with $N = 256$.

Remarks and future work

- The extensions of Fornberg's original method are essentially complete. $I + \textit{compact}$ inner systems carry over.
- (The ellipse method was not presented here.)
- The MATLAB codes need to be refined and integrated.
- Further comparisons with Wegmann's methods needs to be done
- An initial version of the code needs to be publicly available.
- Some additional features and improvements are needed:
  - Add grids from slit maps for Green's, Neumann, and Robin functions.
  - Removal of corners with power maps.
  - Code optimization.
  - Automation for initial guesses.
  - Analytic explanation of the nullspace of the matrix A.