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SUPERLINEAR CONVERGENCE ESTIMATES FOR A CONJUGATE GRADIENT METHOD FOR THE BIHARMONIC EQUATION*

RAYMOND H. CHAN†, THOMAS K. DELILLO‡, AND MARK A. HORN‡

Abstract. The method of Muskhelishvili for solving the biharmonic equation using conformal mapping is investigated. In [R. H. Chan, T. K. DeLillo, and M. A. Horn, *SIAM J. Sci. Comput.*, 18 (1997), pp. 1571–1582] it was shown, using the Hankel structure, that the linear system in [N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen, the Netherlands] is the discretization of the identity plus a compact operator, and therefore the conjugate gradient method will converge superlinearly. Estimates are given here of the superlinear convergence in the cases when the boundary curve is analytic or in a Hölder class.

Key words. biharmonic equation, numerical conformal mapping, Hankel matrices, conjugate gradient method

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1. Introduction.

In [CDH] a method for the solution of boundary value problems for the biharmonic equation using conformal mapping was investigated. The method is an implementation of the classical method of Muskhelishvili [Musk]. In [CDH] it was shown, using the Hankel structure, that the linear system in [Musk] is the discretization of the identity plus a compact operator. In this case, q-superlinear convergence of the conjugate gradient method, where the speed of convergence depends on the right-hand side, has been proven in [Dan]. In the present paper, estimates are given for the decay rates of the eigenvalues of the compact operators when the boundary curve is analytic or in a Hölder class. These estimates are used to give detailed bounds for the r-superlinear convergence which do not depend on the right-hand side.

The paper is organized as follows. Section 2 describes the original method for simply connected regions. The representation of the biharmonic function and the boundary conditions in terms of the analytic Goursat functions is given. Transplanting the boundary conditions to the unit disk with a conformal map then leads to a linear system for the Taylor coefficients of the Goursat functions on the disk. In section 3, some results from conformal mapping are used to show that the linear system can be formulated in terms of a compact operator with a Hankel structure. In section 4, the superlinear convergence rates of the conjugate gradient method applied to the linear system are established.

2. The biharmonic equation.

As in [CDH], we will follow the presentation in [KK] and [Musk]. We wish to find a function $u = u(\eta, \mu)$ which satisfies the

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†Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong (rchan@math.cuhk.hk). The research of this author was partially supported by HKRGC grant CUHK316/94E.

‡Department of Mathematics and Statistics, Wichita State University, Wichita, KS 67260-0033 (delillo@twsuvm.uc.twsu.edu, mhorn@twsuvm.uc.twsu.edu). The research of the second and third authors was partially supported by U. S. Department of Energy grant DE-FG02-92ER25124 and National Science Foundation EPSCoR grant OSR-9255223.

biharmonic equation,

$$\Delta^2 u = 0,$$

for $\zeta = \eta + i\mu \in \Omega$ where Ω is a region with a smooth boundary Γ and u satisfies the boundary conditions

$$u_\eta = G_1 \quad \text{and} \quad u_\mu = G_2$$

on Γ. This boundary value problem arises, for instance, in plane stress problems where u is the so-called Airy stress function. u can be represented as

$$u(\zeta) = \mathrm{Re}(\overline{\zeta}\phi(\zeta) + \chi(\zeta)),$$

where $\phi(\zeta)$ and $\chi(\zeta)$ are analytic functions in Ω known as the *Goursat functions.* Letting $G = G_1 + iG_2$, the boundary conditions become

$$\phi(\zeta) + \zeta\overline{\phi'(\zeta)} + \overline{\psi(\zeta)} = G(\zeta), \quad \zeta \in \Gamma, \tag{1}$$

where $\psi(\zeta) = \chi'(\zeta)$. The problem is to find ϕ and ψ satisfying (1).

Let $\zeta = f(z)$ be the conformal map from the unit disk to Ω, fixing $f(0) = 0 \in \Omega$. Then with $d(z) := f(z)/\overline{f'(z)}$, $\phi(z) := \phi(f(z)), \psi(z) := \psi(f(z))$, and $G(z) := G(f(z))$, equation (1) transplants to the unit disk as

$$\phi(z) + d(z)\overline{\phi'(z)} + \overline{\psi(z)} = G(z), \quad |z| = 1. \tag{2}$$

Let

$$\phi(z) = \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad \psi(z) = \sum_{k=0}^{\infty} b_k z^k.$$

The problem is now to find the a_k's and the b_k's.

For $|z| = 1$, define the Fourier series

$$d(z) := f(z)/\overline{f'(z)} = \sum_{k=-\infty}^{\infty} h_k z^k, \quad G(z) = \sum_{k=-\infty}^{\infty} A_k z^k.$$

Substituting into (2) gives a linear system of equations for the a_k's and b_k's,

$$a_j + \sum_{k=1}^{\infty} k\overline{a}_k h_{k+j-1} = A_j, \quad j = 1, 2, 3, \ldots, \tag{3}$$

$$\overline{b}_j + \sum_{k=1}^{\infty} k\overline{a}_k h_{k-j-1} = A_{-j}, \quad j = 0, 1, 2, \ldots. \tag{4}$$

If (3) is solved for the a_k's, then the b_k's can be easily computed from (4). These systems are derived in [Musk] and [KK] and solved for some simple examples. In [CDH], (3) was truncated after n terms and solved efficiently using the conjugate gradient method. The matrix-vector multiplications can be done in $O(N \log N)$ using fast Fourier transforms (FFTs).

There is a moment condition to be satisfied by the data. After transplantation to the disk, this condition can be stated as $\mathrm{Re}[\int_{|z|=1} G(z)\overline{f'(z)}d\overline{z}] = 0$. Furthermore, ϕ

and ψ are not unique. In short, one needs to specify $a_0 = \phi(0) = 0$ and $\mathrm{Im}(a_1/f'(0)) = 0$, which will be incorporated into the derivations.

It should be noted that if our boundary data corresponds to $G = 0$ then the only possible (nonzero) choice for ϕ is $\phi(z) = Cif(z)$, for some nonzero $C \in \mathbf{R}$. This implies that the null space corresponding to the infinite system in (3) is one-dimensional, and the eigenvector spanning this space is given by $a_k = ic_k, k = 1, 2, 3, \ldots$, where $f(z) = \sum_{k=1}^{\infty} c_k z^k$.

3. Compact operators. As in [CDH], we take real and imaginary parts of equation (3) to get

$$\alpha_j + \sum_{k=1}^{\infty} k(\eta_{k+j-1}\alpha_k + \gamma_{k+j-1}\beta_k) = B_j, \quad j = 1, 2, 3, \ldots, \tag{5}$$

$$\beta_j + \sum_{k=1}^{\infty} k(\gamma_{k+j-1}\alpha_k - \eta_{k+j-1}\beta_k) = C_j, \quad j = 1, 2, 3, \ldots, \tag{6}$$

where we have used the notation $a_k = \alpha_k + i\beta_k$, $h_k = \eta_k + i\gamma_k$, and $A_k = B_k + iC_k$. For visualization purposes, we combine equations (5) and (6) into a doubly infinite matrix equation in which the two sums are combined into a block Hankel matrix composed with a diagonal matrix. (A *Hankel matrix* is a matrix which is constant on the antidiagonals.) In fact, (5) and (6) can be written as

$$(I_\infty + H_{r,\infty}D_\infty)\underline{\alpha} + H_{i,\infty}D_\infty\underline{\beta} = \underline{B},$$

$$(I_\infty - H_{r,\infty}D_\infty)\underline{\beta} + H_{i,\infty}D_\infty\underline{\alpha} = \underline{C},$$

so that

$$\left(\begin{pmatrix} I_\infty & 0 \\ 0 & I_\infty \end{pmatrix} + \begin{pmatrix} H_{r,\infty} & H_{i,\infty} \\ H_{i,\infty} & -H_{r,\infty} \end{pmatrix}\begin{pmatrix} D_\infty & 0 \\ 0 & D_\infty \end{pmatrix}\right)\begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} = \begin{pmatrix} \underline{B} \\ \underline{C} \end{pmatrix}, \tag{7}$$

where $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots)^T$, $\underline{\beta} = (\beta_1, \beta_2, \ldots)^T$, $\underline{B} = (B_1, B_2, \ldots)^T$, $\underline{C} = (C_1, C_2, \ldots)^T$, I_∞ is the infinite identity matrix, $D_\infty = \mathrm{diag}(1, 2, \ldots)$, $H_{r,\infty}$ is an infinite Hankel matrix generated by the η_k, and $H_{i,\infty}$ is an infinite Hankel matrix generated by the γ_k.

Now suppose $(\underline{\alpha}, \underline{\beta})$ represents a solution to (5),(6). Define

$$\underline{x} = \begin{pmatrix} D_\infty^{1/2}\underline{\alpha} \\ D_\infty^{1/2}\underline{\beta} \end{pmatrix}, \quad \underline{r} = \begin{pmatrix} D_\infty^{1/2}\underline{B} \\ D_\infty^{1/2}\underline{C} \end{pmatrix}.$$

Then (7) can be written as

$$(I_\infty + M_\infty)\,\underline{x} = \underline{r}, \tag{8}$$

where M_∞ is given by

$$M_\infty = \begin{pmatrix} M_{r,\infty} & M_{i,\infty} \\ M_{i,\infty} & -M_{r,\infty} \end{pmatrix} = \begin{pmatrix} D_\infty^{1/2}H_{r,\infty}D_\infty^{1/2} & D_\infty^{1/2}H_{i,\infty}D_\infty^{1/2} \\ D_\infty^{1/2}H_{i,\infty}D_\infty^{1/2} & -D_\infty^{1/2}H_{r,\infty}D_\infty^{1/2} \end{pmatrix}.$$

Note that M_∞ is symmetric. We would now like to justify the formal manipulations above and show that M_∞ is a compact operator. This will require the following definitions and lemmas; see, e.g., [Po].

Definition 1. *For $l \geq 1$ and $0 < \alpha \leq 1$, $\gamma \in C^{l,\alpha}[0, 2\pi]$ if γ is l-times differentiable for $0 \leq s \leq 2\pi$ and*

$$|\gamma^{(l)}(s_1) - \gamma^{(l)}(s_2)| \leq C|s_1 - s_2|^{\alpha}.$$

Definition 2. *The Jordan curve Γ is of class $C^{l,\alpha}$ if it has a parameterization $\Gamma : \gamma(s)$, $0 \leq s \leq 2\pi$, such that $\gamma \in C^{l,\alpha}[0, 2\pi]$ and $\gamma'(s) \neq 0$.*

Next, we state a few well-known results about Fourier coefficients and conformal maps.

Theorem 1. *Let f be periodic with Fourier series $f(e^{i\theta}) = \sum_{n=1}^{\infty} c_n e^{in\theta}$. Then*

(i) *f is analytic implies that there is an $R, 0 < R < 1$, such that*

$$|c_n| = O(R^{|n|});$$

(ii) *$f \in C^{l,\alpha}[0, 2\pi]$ implies*

$$|c_n| = O(n^{-l-\alpha}), \quad l \geq 1, \quad 0 < \alpha \leq 1.$$

Warschawski's Theorem. *Let f map the disk D conformally onto the inner domain of the Jordan curve Γ of class $C^{l,\alpha}$, where $l \geq 1$ and $0 < \alpha < 1$. Then $f^{(l)}$ has a continuous extension to $\overline{D}$ and*

$$|f^{(l)}(z_1) - f^{(l)}(z_2)| \leq C|z_1 - z_2|^{\alpha}, \quad z_1, z_2 \in \overline{D}.$$

Note that in general one cannot take $\alpha = 1$ in the above theorem.

Lemma 1. *Let f be a conformal map from the unit disk to the region Ω with boundary Γ. Assume*

$$f(e^{i\theta})/\overline{f'(e^{i\theta})} = \sum_{n=-\infty}^{\infty} h_n e^{in\theta}.$$

Then

(i) *Γ is analytic implies that there is an $r, 0 < r < 1$, such that*

$$|h_n| = O(r^{|n|});$$

(ii) *for $l \geq 2$ and $0 < \alpha < 1$, $\Gamma \in C^{l+1,\alpha}$ implies*

$$|h_n| = O(n^{-l-\alpha}).$$

Proof. The proof of (i) can be found in [CDH].

By Warschawski's Theorem, Γ is of class $C^{l+1,\alpha}$ implies $f(e^{i\theta}) \in C^{l+1,\alpha}[0, 2\pi]$ and $\overline{f'}(e^{i\theta}) \in C^{l,\alpha}[0, 2\pi]$. Note that, since $f(z)$ is conformal and smooth for $|z| \leq 1$, $f'(z) \neq 0$ for $|z| \leq 1$. The proof of (ii) then follows from Theorem 1, since $f/\overline{f'} \in C^{l,\alpha}[0, 2\pi]$ for $l \geq 2$, $0 < \alpha < 1$. □

Theorem 2. *If Γ is analytic or of class $C^{l+1,\alpha}$, $l \geq 2$, $0 < \alpha < 1$, then $M_{r,\infty} : l^1 \to l^1$ and $M_{i,\infty} : l^1 \to l^1$ are compact operators where for $\underline{y} \in l^1$,*

$$M_{r,\infty}\underline{y} = \sum_{k=1}^{\infty} \sqrt{kj}\,\eta_{k+j-1} y_k, \quad j = 1, 2, \ldots,$$

$$M_{i,\infty}\underline{y} = \sum_{k=1}^{\infty} \sqrt{kj}\,\gamma_{k+j-1} y_k, \quad j = 1, 2, \ldots.$$

Proof. We will prove the theorem for $M_{r,\infty}$. The case for $M_{i,\infty}$ follows similarly. Define the finite rank operators $\{M_{r,n}\} = \{D_n^{1/2} H_{r,n} D_n^{1/2}\}$ by

$$M_{r,n}\underline{y} = \sum_{k=1}^{n} \sqrt{kj}\,\eta_{k+j-1} y_k, \quad j = 1, 2, \ldots, n,$$

for all $\underline{y} = (y_1, y_2, \ldots) \in l^1$. The goal is to show that $M_{r,\infty}$ can be approximated uniformly by these finite rank operators. (Then, e.g., a version of Theorem 4.4c [Con, p. 41] for Banach spaces shows that $M_{r,\infty}$ is itself compact.) If $A = (a_{kj})$ is an infinite matrix, then the induced l^1 operator norm is given by

$$||A||_{l^1} = \sup_j \sum_{k=1}^{\infty} |a_{kj}|;$$

see, e.g., [Con, p. 171, prob. 8].

The case when Γ is analytic is given in [CDH], so assume Γ is of class $C^{l+1,\alpha}$. Let $m_{k,j}$ denote the (k,j)th component of $M_{r,n}$. Then clearly, $m_{k,j} = \sqrt{kj}\,\mathrm{Re}(h_{k+j-1})$. From Lemma 1 we obtain the following estimate. For any $j \geq 1$,

$$\begin{aligned}
\sum_{k=1}^{\infty} |m_{k,j}| &\leq C \sum_{k=1}^{\infty} \frac{\sqrt{kj}}{(k+j-1)^{l+\alpha}} \\
&\leq C\sqrt{j} \sum_{k=0}^{\infty} \frac{1}{(k+j)^{l-1/2+\alpha}} \\
&\leq C\sqrt{j} \Big\{ \frac{1}{j^{l-1/2+\alpha}} + \int_0^{\infty} \frac{dx}{(x+j)^{l-1/2+\alpha}} \Big\} \\
&\leq C \frac{1}{j^{l-2+\alpha}},
\end{aligned}$$

where C is a constant that depends only on the conformal map. Consequently,

$$\begin{aligned}
||M_{r,\infty} - M_{r,n}||_{l^1} &= \sup \Big\{ \sum_{k=1}^{\infty} |m_{k,n+1}|,\ \sum_{k=1}^{\infty} |m_{k,n+2}|, \ldots \Big\} \\
&\leq C \sup \Big\{ \frac{1}{(n+1)^{l-2+\alpha}}, \frac{1}{(n+2)^{l-2+\alpha}}, \ldots \Big\} \\
&= C \frac{1}{(n+1)^{l-2+\alpha}}.
\end{aligned}$$

The result follows. $\square$

COROLLARY 1. *Under the notation and assumptions of Theorem* 1, M_∞ *is compact on* $l^1 \times l^1$, *where for* $\underline{x} = (\underline{x}^1, \underline{x}^2) \in l^1 \times l^1$,

$$M_\infty \begin{pmatrix} \underline{x}^1 \\ \underline{x}^2 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{\infty} \sqrt{kj}\,\eta_{k+j-1} {x^1}_k + \sum_{k=1}^{\infty} \sqrt{kj}\,\gamma_{k+j-1} {x^2}_k \\ \sum_{k=1}^{\infty} \sqrt{kj}\,\eta_{k+j-1} {x^1}_k - \sum_{k=1}^{\infty} \sqrt{kj}\,\gamma_{k+j-1} {x^2}_k \end{pmatrix}.$$

The norm on $l^1 \times l^1$ *is given by* $||\underline{x}||_{l^1 \times l^1} = ||\underline{x}^1||_{l^1} + ||\underline{x}^2||_{l^1}$.

Proof. From the notation of the problem, it is easily verified that

$$||M_\infty - M_n||_{l^1 \times l^1} \leq 2(||M_{r,\infty} - M_{r,n}||_{l^1} + ||M_{i,\infty} - M_{i,n}||_{l^1}).$$

The result follows as in Theorem 2. $\square$

Though Corollary 1 shows that M_∞ is compact, a precise estimate on the eigenvalues of M_∞ is needed to obtain superlinear convergence rates for error vectors of the conjugate gradient method. This leads us to the next section.

4. Superlinear convergence rates. First, we will discretize the infinite systems above. As in [CDH], we do this by truncating (3) after n terms, replacing the h_k for $k = 1, \ldots, n$ by the values computed with the discrete Fourier transform, and setting $h_k = 0$ for $k = n+1, \ldots, N-1$, where $N = 2n$. The decay of the h_k's given in Lemma 1 will guarantee that the resulting finite system is only a small perturbation of the infinite system. We denote by $\underline{x}^{(n)}$ and $\underline{r}^{(n)}$ the N-vectors formed by truncating $\underline{x}$ and $\underline{r}$ and by M_n the $N \times N$ matrix formed by truncating M_∞, etc. Then, for instance,

$$M_n = \begin{pmatrix} M_{r,n} & M_{i,n} \\ M_{i,n} & -M_{r,n} \end{pmatrix},$$

where the (k,j)th entry of $M_{r,n}$ and $M_{i,n}$ are, respectively, $\sqrt{kj}\mathrm{Re}(h_{k+j-1})$ and $\sqrt{kj}\mathrm{Im}(h_{k+j-1})$.

Our problem is then to solve the truncated version of (8), the $N \times N$ system

$$(I_n + M_n)\underline{x}^{(n)} = \underline{r}^{(n)}.$$

Recall that $\underline{x}^{(n)}$ is subject to a uniqueness condition. Since $f'(0) > 0$, the condition $\mathrm{Im}(a_1/f'(0)) = 0$ implies $x_{n+1}^{(n)} = 0$.

Put $A_\infty = I_\infty + M_\infty$ and $A_n = I_n + M_n$. Then A_∞ has a one-dimensional null space with null vector

$$\underline{v} = (-\mathrm{Im}\ c_1, -\sqrt{2}\mathrm{Im}\ c_2, \ldots, \mathrm{Re}\ c_1, \sqrt{2}\mathrm{Re}\ c_2, \ldots)^T.$$

Since M_∞ is compact, A_n has numerical rank $N-1$ for large $N = 2n$. Moreover, the corresponding near-null vector is approximated by the truncated version of $\underline{v}$; see [An]. We will assume that A_∞ is positive semidefinite. This will imply that A_n is positive semidefinite for large n. (This is the case in our numerical examples.) But then this implies that A_n is positive definite on $\underline{v}^{(n)\perp}$ and the conjugate gradient will converge in this subspace if the initial guess $\underline{x}_0^{(n)} = \underline{0}$ is chosen.

Our solution can be written in the form

$$\underline{y}^{(n)} = \underline{x}^{(n)} + \delta \underline{v}^{(n)},$$

where δ is to be determined from the uniqueness condition. Thus, the conjugate gradient will be applied to find the $\underline{x}^{(n)}$ in $\underline{v}^{(n)\perp}$. Our goal is to give a precise estimate for the superlinear convergence of the method. Define the norm $\|\underline{x}\|_A^2 = \underline{x}^T A \underline{x}$ and the error vector at the qth step $\underline{e}_q = \underline{x}^{(n)} - \underline{x}_q^{(n)}$. We are now ready for the main result.

THEOREM 3. *Assume A_∞ is positive semidefinite with exactly one null vector $\underline{v}$. Then, for large n, the error vector $\underline{e}_q$ at the qth step of the conjugate gradient method applied to $\underline{v}^{(n)\perp}$ satisfies the following estimates.*

(i) *If Γ is analytic, there is an r, $0 < r < 1$, such that*

$$\|\underline{e}_{4q}\|_{A_n} \le C^q r^{q^2} \|\underline{e}_0\|_{A_n}. \tag{9}$$

(ii) *If Γ is of class $C^{l+1,\alpha}$, $l \ge 2$, $0 < \alpha < 1$, then*

$$\|\underline{e}_{4q}\|_{A_n} \le \frac{C^q}{((q-1)!)^{2(l-2+\alpha)}} \|\underline{e}_0\|_{A_n}. \tag{10}$$

Here C is a constant that depends on the conformal map.

Proof. The proof follows closely the proof of Theorem 3 in [Chan]. From the standard error analysis of the conjugate gradient method, we have

$$\|\underline{e}_q\|_{A_n} \le [\,\min_{P_q} \max_{\lambda\in\sigma(A_n)} |P_q(\lambda)|\,]\,\|\underline{e}_0\|_{A_n}, \tag{11}$$

where the minimum is taken over polynomials of degree q with constant term 1 and the maximum is taken over $\sigma(A_n)$, the spectrum of A_n; see, for instance, [GVL]. Actually, we restrict A_n to $\underline{v}^{(n)\perp}$ as discussed above so that by [An], for large n, $\sigma(A_n) \subset (\epsilon, \infty)$ for some $\epsilon > 0$ which is independent of n. In the following, we will try to estimate the minimum in (11). We will prove (9) first.

Assume Γ is analytic. We write

$$M_n = W_n^{(k)} + U_n^{(k)} \quad \forall k \ge 1, \tag{12}$$

where $U_n^{(k)}$ is the matrix obtained from keeping the number $1, 2, \ldots, k$ and number $n+1, n+2, \ldots, n+k$ columns and rows of M_n while replacing all other entries of M_n by zeros. Clearly, rank $(U_n^{(k)}) \le 4k$. By Lemma 2 of [CDH], for all $k \ge 1$, $W_n^{(k)}$ with $r = \rho^2$ satisfies

$$\|W_n^{(k)}\|_1 \le \sup_{k+1\le j\le n} \sum_{i=k+1}^{n} \sqrt{ij}\,|h_{i+j-1}| \le \sup_{k+1\le j\le n} \sum_{i=k+1}^{n} c\rho^{i+j} \le cr^k.$$

Note that $W_n^{(k)}$ is symmetric since M_n is symmetric. Moreover, for any symmetric matrix we have $\|A\|_2 \le \|A\|_1$. It follows that

$$\|W_n^{(k)}\|_2 \le cr^k \quad \forall k \ge 1. \tag{13}$$

Let us order the eigenvalues of $M_n =: W_n^{(0)}$ as

$$\mu_0^- \le \mu_{1/2}^- \le \mu_1^- \le \mu_{3/2}^- \le \cdots \le \mu_{3/2}^+ \le \mu_1^+ \le \mu_{1/2}^+ \le \mu_0^+.$$

By applying the Cauchy interlace theorem [GVL] to (12) and using the bound of $\|W_n^{(k)}\|_2$ in (13), we see that

$$|\mu_k^\pm| \le cr^{\lfloor k \rfloor} \quad \forall \lfloor k \rfloor \ge 1.$$

Thus if we order the eigenvalues of $A_n = I_n + W_n^{(0)}$ as

$$0 < \lambda_0^- \le \lambda_{1/2}^- \le \lambda_1^- \le \lambda_{3/2}^- \le \cdots \le \lambda_{3/2}^+ \le \lambda_1^+ \le \lambda_{1/2}^+ \le \lambda_0^+,$$

then $\lambda_k^\pm = 1 + \mu_k^\pm$ for all $k \ge 0$ with

$$1 - cr^{\lfloor k \rfloor} \le \lambda_k^- \le \lambda_k^+ \le 1 + cr^{\lfloor k \rfloor} \quad \forall k \ge 1. \tag{14}$$

Having obtained the bounds for $\lambda_k^\pm$, we can now construct the polynomial that will give us a bound for (11). Our idea is to choose a P_{4q} that annihilates the $2q$ extreme pairs of eigenvalues. Thus consider

$$p_k(x) = \left(1 - \frac{x}{\lambda_k^+}\right)\left(1 - \frac{x}{\lambda_{k+1/2}^-}\right)\left(1 - \frac{x}{\lambda_{k+1/2}^+}\right)\left(1 - \frac{x}{\lambda_k^-}\right) \quad \forall k \ge 0.$$

Between the roots $\lambda_k^{\pm}$, the maximum of $|(1-\frac{x}{\lambda_k^-})(1-\frac{x}{\lambda_k^+})|$ is attained at the average $x=\frac{1}{2}(\lambda_k^+ + \lambda_k^-)$. Consequently,

$$\begin{aligned}\max_{x\in[\lambda_{k+1/2}^-,\lambda_{k+1/2}^+]}|p_k(x)| &\le \frac{(\lambda_k^+-\lambda_k^-)^2}{4\lambda_k^+\lambda_k^-}\frac{(\lambda_{k+1/2}^+-\lambda_{k+1/2}^-)^2}{4\lambda_{k+1/2}^+\lambda_{k+1/2}^-}\\ &\le \frac{(\lambda_k^+-\lambda_k^-)}{4\lambda_k^-}\frac{(\lambda_{k+1/2}^+-\lambda_{k+1/2}^-)}{4\lambda_{k+1/2}^-}\\ &\le \frac{(\lambda_k^+-\lambda_k^-)^2}{16(\lambda_0^-)^2}\quad \forall k\ge 0.\end{aligned}$$

But then by (14) we have

$$\max_{x\in[\lambda_{k+1/2}^-,\lambda_{k+1/2}^+]}|p_k(x)| \le \frac{(\lambda_k^+-\lambda_k^-)^2}{16(\lambda_0^-)^2}\le Cr^{2k}\quad \forall k\ge 1. \tag{15}$$

Hence the polynomial $P_{4q}=p_0p_1\cdots p_{q-1}$, which annihilates the $2q$ extreme pairs of eigenvalues, satisfies

$$|P_{4q}(x)|\le C^q r^{q^2} \tag{16}$$

for all $x=\lambda_k^{\pm}$ in the inner interval between $\lambda_{q-1/2}^-$ and $\lambda_{q-1/2}^+$, where the remaining eigenvalues are. Here C is a constant that depends only on f. It should be noted that, due to the one-dimensional null space of the compact operator M_∞, we must have λ_0^- uniformly bounded away from zero for large N [An, Thm. 4.8]. Thus, C is uniformly bounded for large N. (9) now follows directly from (11) and (16).

The idea for the estimate (10) is essentially the same with only minor modifications. Assume Γ is of class $C^{l+1,\alpha}$, where $l\ge 2, 0<\alpha<1$. $W_n^{(k)}$ satisfies

$$\begin{aligned}\|W_n^{(k)}\|_1 &\le \sup_{k+1\le j\le n}\sum_{i=k+1}^{n}\sqrt{ij}|h_{i+j-1}|\\ &\le \sup_{k+1\le j\le n}\sum_{i=k+1}^{n}\frac{c\sqrt{ij}}{(i+j-1)^{l+\alpha}}\\ &\le \sup_{k+1\le j\le n}(\sqrt{j})\int_k^\infty \frac{c\,dx}{(x+j-1)^{l-1/2+\alpha}}\\ &\le \frac{c}{k^{l-2+\alpha}}.\end{aligned}$$

Using the previous labeling convention, (14) becomes

$$1-\frac{c}{\lfloor k\rfloor^{l-2+\alpha}}\le\lambda_k^-\le\lambda_k^+\le 1+\frac{c}{\lfloor k\rfloor^{l-2+\alpha}}\quad\forall k\ge 1.$$

And so (15) becomes

$$\max_{x\in[\lambda_{k+1/2}^-,\lambda_{k+1/2}^+]}|p_k(x)| \le \frac{(\lambda_k^+-\lambda_k^-)^2}{16(\lambda_0^-)^2}\le \frac{C}{k^{2(l-2+\alpha)}}\quad \forall k\ge 1.$$

Finally, we obtain the estimate

$$(17) \qquad |P_{4q}(x)| \leq \frac{C^q}{((q-1)!)^{2(l-2+\alpha)}}.$$

The estimate (10) now follows directly from (11) and (17). $\square$

In case A_∞ is not semidefinite we can solve the normal equations by the conjugate gradient method. It is clear that $(I_n + M_n)^2$ will then be positive definite on $\underline{v}^{(n)\perp}$. Using the techniques in the proof of Theorem 3 one can establish a similar result for the normal equations. This will not be done here. In [CDH], we solved the normal equations. However, in all of our examples so far, we have found that $I_n + M_n$ is (nearly) positive semidefinite, so that it is sufficient to just solve $(I_n + M_n)\underline{x}^{(n)} = \underline{r}^{(n)}$ by conjugate gradient. The numerical results are similar to those reported in [CDH]. For a discussion of the numerical conformal mapping methods used, also see [CDH].

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