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ITERATIVE REGULARIZATION METHODS FOR INVERSE PROBLEMS IN ACOUSTICS

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ABSTRACT

We consider the use of conjugate-gradient-like iterative methods for the solution of integral equations arising from an inverse problem in acoustics in a bounded three dimensional region. The inverse problem is the computation of the normal velocities on the boundary of a region from pressure measurements on an interior surface. The pressure satisfies the Helmholtz equation in the region. Two formulations are considered: one based on the representation of pressures by a single layer potential and the other based on the Helmholtz-Kirchhoff integral equation. Both formulations can be used to approximate the Neumann Green's function as an alternative. The integral equations are all ill-posed and are discretized by a boundary element method. The resulting linear systems are ill-conditoned and a (smooth) regularized solutions must be sought. Two regularization rules, including a new one, for conjugate-gradient-like methods are applied and found to have advantages over a standard method based on the truncated singular value decomposition using generalized cross validation. Due to the occurence of multiple singular values for our integral operators, conjugate gradient methods compute the optimal solution in the first few iterations and prove to be particularly fast for these large scale acoustics problems.

INTRODUCTION

In the last two decades, the method of reconstructing the acoustical field and the normal velocities of a vibrating structure using pressure measurements from an array of microphones in the nearfield, known as nearfield acoustical holography (NAH), has become a standard technique of acousticians seeking to locate sources of noise produced by aircraft, automobiles, and other machinery; see [14] for an introduction to this field. The standard mathematical formulations of this inverse problem are in terms of first-kind integral equations for the Helmholtz equation and are ill-posed. The numerical solution of the integral equations therefore requires regularization techniques to filter the effects of limited accuracy in the pressure measurements. During the last two decades a great deal of theoretical and numerical work has also been done on inverse problems, in general; see e.g. [8], [9], [10], [11]. In [4], [5], [6], [7], the present authors and their colleagues have formulated and investigated the use of a single layer potential representation for solutions to the Helmholtz equation for inverse problems in two and three dimensions. The present paper extends this work to include the Helmholtz-Kirchhoff integral equation, the Neumann-Green's function (see, e.g., [16] for an example of a practical computation), and the application of a new stopping rule for regularization using the conjugate gradient method [4]. In the following sections, we will recall the mathematical formulation, review the numerical and regularization methods, and report on computations for two geometries, the interior of a sphere and a fuselage, using an exterior point source as a test case.

MATHEMATICAL FORMULATION

We consider the problem of identifying the source of the acoustical noise on the surface $\Gamma = \partial D$ of an $\mathbb{R}^3$ domain D. The acoustical field p of frequency k in D satisfies the Helmholtz equation

$$\Delta p + k^2 p = 0, \quad \text{in } D \tag{1}$$

In our applications D represents the cabin of an aircraft and Γ represents the fuselage. Acoustical sensors are located on a surface Γ_0 inside the cabin. These sensors measure the field p and the problem is to recover from these measurements the so-called normal velocity

$$v = \frac{\partial p}{\partial \nu}, \quad \text{in } \Gamma = \partial D, \tag{2}$$

where ν is the unit exterior normal to Γ.

The approaches used to solve this inverse problem will depend on the representation of the solution to (1) and (2). The first approach, using a single layer potential to represent the acoustical pressure, was discussed in [6]. The second is a popular approach based on the representation of the pressure as a combination of single and double layer potentials. The solution then will be given by solving the Helmholtz-Kirchhoff system of integral equations. Both approaches can be used to derive the Neumann Green's function. Theoretical results on existence, uniqueness, and stability are discussed in [5], [6], and [12].

The Single Layer approach

In [5] and [6] it was shown that p can be represented in D by a single layer potential

$$p(x) = (S\varphi)(x) := \int_\Gamma \Phi(x,y)\,\varphi(y)\,dS(y) \quad x \in D. \tag{3}$$

The inverse problem is reduced to finding the density φ, which solves the Fredholm integral equation of the first kind

$$\int_\Gamma \Phi(x,y)\,\varphi(y)\,dS(y) = p(x), \quad x \in \Gamma_0. \tag{4}$$

The normal velocity will be obtain using the density φ in the integral equation

$$v(x) = (S'\varphi)(x) := \varphi(x)/2 + \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(x)}\varphi(y)\,dS(y) \tag{5}$$

$x \in \Gamma$, which follows from (2) and the jump relations for the normal derivative of single layer potentials.

Equation (5) will allow us to obtain explicitly the Neumann Green's function

$$\left(\mathcal{G}^2\varphi\right)(x) := \left(S\,(S')^{-1}\varphi\right)(x),$$

by inverting the well-posed operator S'. The solution of the inverse problem will be reduced to the solution of the integral equation

$$\left(\mathcal{G}^2 v\right)(x) = p(x), \quad x \in \Gamma_0. \tag{6}$$

The Helmholtz-Kirchhoff Approach

The classical approach (see [3]) uses the representation

$$p(x) = (Sv)(x) - (Dp)(x), \quad x \in D, \tag{7}$$

where

$$(D\varphi)(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\,dS(y)$$

is the double layer potential and

$$\Phi(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|},$$

is the free space radiating fundamental solution to the Helmholtz equation. Using the jump relations for the double layer potential we obtain

$$\frac{p(x)}{2} = \left(S^- v\right)(x) - \left(D^- p\right)(x), \quad x \in \Gamma, \tag{8}$$

where S^- and D^- denote the single and double layer operator respectively in Γ.

When D is a separable geometry, we can obtain an explicit formula for the Neumann Green's function of this representation (see [14].) For general domains, equation (8) will allow us to obtain the Neumann Green's function

$$\left(\mathcal{G}^1\varphi\right)(x) := (S\varphi)(x) - \left(D\left(\frac{1}{2}I + D^-\right)^{-1} S^-\varphi\right)(x),$$

by inverting the well-posed operator $\left(\frac{1}{2}I + D^-\right)$. The solution of the inverse problem will be reduced to the solution of the integral equation

$$\left(\mathcal{G}^1 v\right)(x) = p\,(x)\,, \quad x \in \Gamma_0. \tag{9}$$

A different but close related approach will be to solve the Helmholtz-Kirchhoff system

$$\begin{pmatrix} Sv & -Dp \\ S^- v & -D^- p - \frac{1}{2}p \end{pmatrix}(x) = \begin{pmatrix} p \\ 0 \end{pmatrix}(x)\,, \quad x \in \Gamma_0, \tag{10}$$

for v and p on Γ. In principle this method will avoid the explicit inversion of any operator and will be useful for iterative methods.

NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS AND REGULARIZATION METHODS

We consider the numerical solution of the integral equations (9), (10), (4) and (6) for a range of wave numbers k. Recall that $k = \omega/c$, where ω denotes the frequency and c is the speed of sound. In the case of the interior aircraft cabin noise $c = 340\ m/\sec$ and the range of interest of ω is typically in the range of human speech

$$20\pi/\sec < \omega = 2\pi f < 1000\pi/\sec,$$

where f is the frequency in Hertz. This leads to a range of k of

$$0.06\pi/m < k < 3\pi/m.$$

Since the cabin dimensions a are on the order of meters, the dimensionless quantity ka has a similar range.

Numerical Algorithms

All the programing has been done in MATLAB and run on 900 MHz pentium PCs with 512 MB of RAM memory. The different approaches for the solution to the inverse acoustical problem require the numerical solution of integral equations. In [5] a Nyström method was used for the discretization of S and S' in the two dimensional space. For our case of three dimensions we apply the boundary element method described in [6], [7] [1], [2] to discretize the operators S, S', D, D^- and S^- into $N \times N$ complex matrices S_N, S'_N, D_N, D^-_N and S^-_N.

The single layer approach will require us to solve equation (4) for the density φ and then use (5) to recover the normal velocity. The numerical method is reduced to the solution of the linear system

$$S_N \widetilde{\varphi} = \widetilde{p} \tag{11}$$

where $\widetilde{\varphi}$ and $\widetilde{p}$ are $N \times 1$ vectors. The numerical calculation of the acoustic velocity on Γ is given by

$$\widetilde{v} = S'_N \widetilde{\varphi}, \tag{12}$$

where $\widetilde{v}$ is an $N \times 1$ vector. For the purpose of labeling in the numerical methods, we will call this method S_N, S'_N.

Another method based on the single layer representation will be to use equation (6). This equation will be reduced to the linear system

$$\mathcal{G}^2_N \widetilde{v} = \widetilde{p}, \tag{13}$$

where $\mathcal{G}^2_N$ is the $N \times N$ matrix defined as

$$\mathcal{G}^2_N = S_N \left(S'_N\right)^{-1}.$$

For the representation (7) one method for computing the solution will be to solve equation (9), which will be reduced to the solution of the linear system

$$\mathcal{G}^1_N \widetilde{v} = \widetilde{p}, \tag{14}$$

where $\mathcal{G}^1_N$ is the $N \times N$ matrix defined as

$$\mathcal{G}^1_N = S^-_N - D_N \left(\frac{1}{2}I_N + D^-_N\right)^{-1} S^-_N.$$

The other method will be to solve the discretization of system (10)

$$\mathcal{H}_N \begin{pmatrix} \widetilde{v} \\ \widetilde{p}^- \end{pmatrix} = \begin{pmatrix} \widetilde{p} \\ 0 \end{pmatrix}, \tag{15}$$

where

$$\mathcal{H}_N = \begin{pmatrix} S_N & -D_N \\ S^-_N & -D^-_N - \frac{1}{2}I_N \end{pmatrix}$$

is a $2N \times 2N$ matrix, 0 is an $N \times 1$ vector, and $\widetilde{p}^{-}$ is the $N \times 1$ vector for the pressure on the boundary Γ.

In a real situation the data $\widetilde{p}$ will contain errors and we will denote

$$\widetilde{p}^{\delta} = \widetilde{p} + e,$$

where e is an $N \times 1$ random vector with $\|e\|_2 = \delta \|\widetilde{p}\|_2$ and δ is the percentage of error. It is well known that the discretized linear systems (11), (13), (14) and (15) are ill-posed, i.e. the presence of errors in $\widetilde{p}^{\delta}$ will be amplified in the solution $\widetilde{v}$, and in most cases the recovery of $\widetilde{v}$ will be useless. Hence regularization methods for (11), (13), (14) and (15) are required.

Numerical Regularization

Consider the solution of the ill-posed linear matrix system

$$A\widetilde{x} = \widetilde{p}, \tag{16}$$

where $\widetilde{x}$, $\widetilde{p}$ are $N \times 1$ vectors and A is an $N \times N$ complex matrix. A standard numerical implementation of regularization methods is based on the Singular Value Decomposition (SVD). The SVD of A will give a decomposition of the form

$$A = U\Sigma V^{*} = \sum_{i=1}^{N} u_i \sigma_i v_i^{*} \tag{17}$$

where $U = (u_1, ..., u_N) \in \mathbb{C}^{N\times N}$ and $V = (v_1, ..., v_N) \in \mathbb{C}^{N\times N}$ are matrices with orthonormal columns, $U^{*}U = V^{*}V = I_N$, and where the diagonal matrix $\Sigma = diag\,(\sigma_1, ..., \sigma_N)$ has nonnegative diagonal elements appearing in nonincreasing order. In our case $\sigma_N > 0$ and so $\mathrm{rank}(A) = N$.

The decay rate of the singular values of operators determine the ill-posedness of the resulting linear systems. (Note that the singular values of, for instance, S_N approximate the first N singular values of S and $\lim_{N\to 0} \sigma_N = 0$.) The system is mildly ill-posed if $\sigma_n = O\left(n^{-\alpha}\right)$ for $\alpha > 0$, and severely ill-posed otherwise (see [8]). In [6] it was found that for the sphere of radius r less than 1, the singular values for S_N had the asymptotic behavior $\sigma_n = O\left(r^n \left(2n+1\right)^{-1}\right)$. This result shows that the system in (11) is ill-posed. Similar results can be found for the systems (13), (14) and (15).

It is instructive to recall the (least squares) solution of (16) using the SVD,

$$\widetilde{x}_{LS} = \sum_{i=1}^{N} \frac{u_i^{*}\widetilde{p}}{\sigma_i} v_i, \tag{18}$$

where in the case of ill-posed linear systems $|u_i^{*}\widetilde{p}|$ decreases with the same rate as σ_i. If we use the representation (18) for the noisy pressure $\widetilde{p}^{\delta}$ then we can expand the series in two terms

$$\widetilde{x}_{LS}^{\delta} = \sum_{i=1}^{N} \frac{u_i^{*}\widetilde{p}}{\sigma_i} v_i + \sum_{i=1}^{N} \frac{u_i^{*}e}{\sigma_i} v_i. \tag{19}$$

Large i correspond to high frequency components. The first term in (19) corresponds to the noiseless solution, while the second term is the error term. The error term will have the property that $|u_i^{*}e|$ will be of roughly the same magnitude δ for all i (see [11]). Due to the decay of the singular values of S_N, the quantity $|u_i^{*}e| / \sigma_i$ will increase significantly with i, affecting the solution. A regularization method is used to reduce the effect of the high frequency components of e term in the solution.

In the next subsections we will describe the regularization methods used in the numerical experiments of this paper.

Truncated Singular Value Decomposition The Truncated Singular Value Decomposition (TSVD) solution $\widetilde{x}_l$, with regularization parameter l, of (16) using the SVD is given by

$$\widetilde{x}_l = \sum_{i=1}^{l} \frac{u_i^{*}\widetilde{p}^{\delta}}{\sigma_i} v_i, \qquad 1 \le l \le rank\,(A)\,. \tag{20}$$

We can see clearly that in the TSVD solution the truncation level l avoids the small singular values that can affect the solution in the presence of noise.

Conjugate Gradient for the Normal Equations The conjugate gradient method is known as one the most powerful algorithms for the solution of the selfadjoint, positive definite well-posed linear equations. We will be interested in the iterative solution of the normal equations of an ill-posed problem. This is called the Conjugate Gradients for the Normal Equations (CGNE).

CGNE for solving equation (16) will be implemented by the iteration for $l = 1, 2, ...$

$$\begin{aligned} \alpha_l &= \left\| A^* r^{(l-1)} \right\|_2^2 / \left\| A d^{(l-1)} \right\|_2^2, \\ \widetilde{x}_l &= \widetilde{x}_{l-1} + \alpha_l d^{(l-1)}, \\ r^{(l)} &= r^{(l-1)} - \alpha_l A d^{(l-1)}, \\ \beta_l &= \left\| A^* r^{(l)} \right\|_2^2 / \left\| A^* r^{(l-1)} \right\|_2^2, \\ d^{(l)} &= A^* r^{(l)} + \beta_l d^{(l-1)}, \end{aligned} \tag{21}$$

where $r^{(l)}$ is the residual vector $r^{(l)} = \widetilde{p}^\delta - A\widetilde{x}_l$ and $d^{(l)}$ is an auxiliary N vector. The CGNE algorithm is initialized with the starting vector $\widetilde{x}_0$ (that can be 0), residual $r^{(0)} = \widetilde{p}^\delta - A\widetilde{x}_l$, and $d^{(0)} = A^* r^{(0)}$.

One of the main advantages of CGNE is that it does not require the computation of the SVD, which is expensive (of order N^3). Rather, the computational cost is $O(N^2)$ per iteration step for multiplication by A and A^*. For CGNE the regularization parameter l will be the number of iterations. For our problems $l << N$ for the optimal solution, making conjugate gradient particularly efficient.

Methods for the Choice of Regularization Parameter

In all regularization methods, the selection of the correct regularization parameter is crucial. In the case of TSVD, an l close to the rank of A will give the least squares solution and a small l will eliminate all the small singular values and will miss important features of the solution. A similar phenomena, called "semiconvergence", occurs with CGNE. The solution of CGNE will approximate the exact solution after a few iterations and further iterations will approximate the least squares solution (19); see Figures 1, 2, 4.

In this section we will present a some methods for the selection of the regularization parameter.

The Discrepancy Principle In 1966, Morozov suggested the discrepancy principle (DP) as means for choosing the regularization parameter in Tikhonov regularization (see [8]). His idea was to allow an error of magnitude δ in the data fit of the computed approximation to prevent unwanted magnification of noise components of the right-hand side. A good estimate of δ is required.

For TSVD and CGNE we terminate at iteration $l\left(\delta, \widetilde{p}^\delta\right)$ when, for the first time, $\left\| A\widetilde{x}_l^\delta - \widetilde{p}^\delta \right\| \leq \tau\delta$, for $\tau > 1$.

Hanke and Raus Method As the previous method, this method will be based on the estimation of the error $\left\| \widetilde{x} - \widetilde{x}_\alpha^\delta \right\|_2$. For CGNE, Hanke and Raus [9] found the estimate

$$\left\| \widetilde{x} - \widetilde{x}_l^\delta \right\|_2 \approx |P_l|^{1/2} \left\| A\widetilde{x}_l - \widetilde{p}^\delta \right\|_2. \tag{22}$$

The quantity P_l will be found using the recursion

$$P_l = \left(\frac{\alpha_l \beta_{l-1}}{\alpha_{l-1}} + 1 \right) P_{l-1} - \frac{\alpha_l \beta_{l-1}}{\alpha_{l-1}} P_{l-2} + \alpha_l, \tag{23}$$

with initial parameters $P_{-1} = P_0 = 0$. The quantities α_j and β_j, $j = 0, 1, 2, ...$ are taken from the recursion (21) and lead to the following stopping rule:
Compute

$$\eta^{HR}(0) = \left\| \widetilde{p}^\delta \right\|_2, \quad \eta^{HR}(l) = |P_l|^{1/2} \left\| A\widetilde{x}_l - \widetilde{p}^\delta \right\|_2, \quad k \geq 1, \tag{24}$$

and terminate the iteration after l_m steps, provided $\eta^{HR}(l_m) \leq \eta^{HR}(l)$, for all $l \geq 0$.

Note that $\eta^{HR}(l)$ is conveniently computed within the conjugate gradient iteration and an estimate of δ is not needed.

DeLillo and Hrycak method [4] presents a parameter choice strategy for the conjugate gradient method which does not require knowledge of δ. Conjugate gradient for the normal equations $A^*Ax = A^*p$ is implemented using Lanczos bidiagonalization [11] with reorthogonalization. The regularization imitates TSVD on the Krylov subspaces. We list the pseudo-code for the computation of the approximate solution vectors, $x_1, \ldots, x_k$. The optimal parameter choice is l such that $\eta^{DH}(l)$ is minimized:

$\quad\beta = \|p\|$
$\quad u = p/\beta$
$\quad q = A' * p$
$\quad a = \|q\|$
$\quad q = q/a$
$\quad Q(:, 1) = q$
$\quad u = A * Q(:, 1)$
$\quad T(1, 1) = \|u\|$
$\quad U(:, 1) = u/T(1, 1)$
$\quad y_1 = (U(:, 1)' * p)/T(1, 1)$
$\quad x_1 = Q(:, 1) * y_1$
$\quad \eta^{DH}(1) = |y_1|$
for $i = 2 : k$
$\quad q = A' * U(:, i-1) - T(i-1, i-1) * Q(:, i-1)$
$\quad q = q - Q(:, 1 : i-1) * (Q(:, 1 : i-1)' * q)$ (reorthogonalize)
$\quad T(i-1, i) = \|q\|$
$\quad Q(:, i) = q/T(i-1, i)$

$u = A * Q(:,i) - T(i-1,i) * U(:,i-1)$
$u = u - U(:,1:i-1) * (U(:,1:i-1)' * u$ (reorthogonalize)
$T(i,i) = \|u\|$
$U(:,i) = u/T(i,i)$
$[UT, ST, VT] = \mathrm{svd}(T(1:i,1:i))$
$y_i = ST \backslash (UT' * U(:,1:i)' * p)$
$x_i = Q(:,1:i) * VT * y_i$
$\eta^{DH}(i) = |y_i|$
end

Generalized Cross-Validation Generalized cross-validation(GCV) is a popular and successful method for choosing the regularization parameter without requiring an estimate of δ. The GCV method is based on statistical considerations, namely, that a good value of the regularization parameter should predict missing data values (see [13]).

The GCV method is a predictive method which seeks to minimize predictive mean-square error $\|A\widetilde{x}_\alpha - \widetilde{p}\|$. Since $\widetilde{p}$ is unknown, the GCV method works instead with the GCV function

$$G(l) = \frac{\left\|A\widetilde{x}_l - \widetilde{p}^\delta\right\|^2}{trace\,(I - AR_l)^2}, \tag{25}$$

where R_l is the operator of the regularizer. The optimal regularization parameter will be the minimum of G.

For TSVD with discrete parameter l, we obtain explicitly $G(l)$ using

$$trace\,(I - AR_l) = N - l. \tag{26}$$

There is not a simple representation of the denominator for CGNE (see [11] for references). We will not use this method with CGNE.

NUMERICAL EXAMPLES

We now give the results of our numerical calculations for two different geometries and various values of k to demonstrate the regularization techniques. Specifically, we will use as our test case the exact acoustical pressure and normal velocity given by a point source located at z,

$$p(x) = \frac{\exp(ik|x-z|)}{4\pi|x-z|}, \qquad x \in \Gamma_0, \tag{27}$$

$$v(x) = \frac{\partial}{\partial\nu(x)}\left(\frac{\exp(ik|x-z|)}{4\pi|x-z|}\right), \qquad x \in \Gamma, \tag{28}$$

where $z \in \mathbb{R}^3 \backslash \overline{D}$. We will denote by $\widetilde{v}$ and $\widetilde{p}$ the $N \times 1$ vectors of the exact pressure and normal velocity calculated using (27) and (28) respectively. Our tests will be based on the recovery of the known normal velocity $\widetilde{v}$ by solving the linear systems (11), (13), (14) and (15) with the acoustical pressure $\widetilde{p}^\delta$. The recovered normal velocity for parameter l is denoted by $\widetilde{v}_l^\delta$.

The Sphere

The surface Γ will be the unit sphere, Γ_0 is the sphere with radius 0.9 and the source is located at $z = (0,2,0)$. We use a BEM [6] with piecewise linear elements which gives a discretization error of $O\left(h^2\right)$ with $h = O\left(1/\sqrt{N}\right)$. The time required to generate the operators $\mathcal{H}_N$ and $\mathcal{G}_N^1$ for $N = 1026$ is approximately 25 minutes, while the time required to generate S_N, S_N' and $\mathcal{G}_N^2$ for the same N is approximately 15 minutes. This is one advantage of using the single layer representation.

In Tables 1 and 2, we show the optimal relative errors $\left\|\widetilde{v}_l^\delta - \widetilde{v}\right\| / \|\widetilde{v}\|$ for different noise levels δ for TSVD and CGNE regularization for the four linear systems used to find the normal velocity for k=1, 3, 6.

The first thing we encounter is that for different noise levels δ, the method of solving for S_N and S' will give larger relative errors than the other methods for smaller wave number k. As we increase k, this method will have smaller relative errors than the other methods. Notice that TSVD regularization will give a smaller relative error for different noise levels δ in the four methods. The relative error of the CGNE solutions is close to the relative error of TSVD.

Although we show that TSVD regularization will give the smaller relative errors, it requires the computation of the SVD which is expensive. For this particular example the computation of the SVD will take about 4 minutes while the computation of 100 CGNE iterations will take approximately 2 minutes. For noisy data we usually require just a few iterations to obtain the optimal solution (see Table 2). Note, in particular, that in all our examples, the optimal l for the iterative methods is small compared with TSVD. This behavior is due the favorable grouping of the singular values for the integral operators of acoustics and is discussed further in [4] and [6].

Figure 1 compares methods for choosing the regularization parameter for TSVD regularization. We plot the function $\log(G(l))$ with the relative error function. Note that DP and GCV have a slight tendency to oversmooth the solution. Still the chosen regularization parameters will give an error close to the optimal, since the choice of regularization parameter is not as sensitive as the case of CGNE.

Figure 2 compares methods for choosing the regularization parameter for CGNE regularization. We plot the functions $\eta^{HR}(l)$ and $\eta^{DH}(l)$ with the relative error function.

Figure 1. TSVD regularization using GCV(O) and DP(∇) to estimate the optimal solution(Δ) for the sphere.

Figure 2. CGNE regularization using HR(O), DH(O) and DP(∇) to estimate the optimal solution(Δ) for the sphere.

Figure 3. View of the boundary triangulation of the fuselage Γ

Here HR and DP tend to oversmooth the solution, while DH slightly undersmooths the solution. It is important to remark that for CGNE, DP will require a good approximation of the noise level. For CGNE the errors are more sensitive to an underestimate or overestimate of the noise level.

The Fuselage

The surface Γ will be a cylinder of radius 1 and length 5 with flat endcaps and a floor at a distance 0.74 from the center. The boundary triangulation with 2048 boundary elements and 1026 nodes is show in Figure 3. This geometry approximately models a Cessna 650 cylindrical fuselage with floor. Γ_0 will be Γ with dimensions reduced by a ratio of 0.9. The source is located at $z = (0, -1.5, -0.5)$.

For the error of the recovery of the regularized solutions we don't consider the corners and edges of the fuselage, because the recovered velocities are not expected to be accurate. In Tables 3 and 4 we show the relative errors of TSVD and CGNE regularization for the four linear systems used to find the normal velocity and $k = 1, 3, 6$.

In contrast to the example of the unit sphere, we observe that, for different noise levels δ, the single layer method S_N, S'_N gives smaller relative errors compared with the other methods for all wave numbers.

Figure 4 compares the methods for choosing the regularization parameter for CGNE regularization for the single layer approach of S_N, S'_N for various noise levels δ. Observe that HR, DH and DP oversmooth the solution, but DH is closer to the optimal solution. As in the unit sphere the parameter chosen by HR is the smallest, the parameter chosen by DP is next largest, and the parameter chosen by DH is the largest.

Figure 4. CGNE regularization using HR(O), DH(O) and DP(∇) to estimate optimal solutions(Δ) for S_N, S'_N for the fuselage for various δ.

CONCLUSIONS

We have demonstrated that conjugate-gradient-like iterative methods methods, as compared to traditional methods such as the truncated singular value decompostion, can be effective and particularly efficient for the integral equations based on single layer, Helmholtz-Kirchhoff, and Green's function formulations of near-field acoustical holography problems. Regularized solutions can be computed in a few iterations with little dependence on the size of the linear systems or the wave number. A new stopping rule [4] for the Lanczos-based iteration was tested and shown to be advantageous. In future work, we plan more extensive comparisons of these methods with other methods such as Tikhonov regularization, the Landweber iteration, and GMRES (see also [15]).

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Method	$k\backslash\delta$	$0.001(l)$	$0.005(l)$	$0.01(l)$	$0.05(l)$
$\mathcal{H}_N$	1	0.059(1203)	0.078(1094)	0.085(1094)	0.136(1064)
$\mathcal{G}_N^1$	1	0.059(178)	0.078(68)	0.085(68)	0.136(38)
S_N, S_N'	1	0.074(198)	0.099(126)	0.105(69)	0.140(37)
$\mathcal{G}_N^2$	1	0.056(198)	0.075(84)	0.081(69)	0.120(38)
$\mathcal{H}_N$	3	0.054(1289)	0.066(1204)	0.079(1125)	0.100(1076)
$\mathcal{G}_N^1$	3	0.054(264)	0.066(178)	0.079(99)	0.101(70)
S_N, S_N'	3	0.051(322)	0.064(150)	0.071(99)	0.106(63)
$\mathcal{G}_N^2$	3	0.051(351)	0.066(173)	0.079(99)	0.109(49)
$\mathcal{H}_N$	6	0.059(1402)	0.070(1290)	0.076(1203)	0.095(1129)
$\mathcal{G}_N^1$	6	0.059(376)	0.070(264)	0.076(166)	0.096(103)
S_N, S_N'	6	0.048(391)	0.057(198)	0.062(150)	0.074(98)
$\mathcal{G}_N^2$	6	0.051(392)	0.064(264)	0.073(150)	0.093(103)

Table 1. $\left\|\widetilde{v}-\widetilde{v}_l^\delta\right\| / \|\widetilde{v}\|$ of TSVD regularization for the sphere.

Method	$k\backslash\delta$	$0.001(l)$	$0.005(l)$	$0.01(l)$	$0.05(l)$
$\mathcal{H}_N$	1	0.059(34)	0.076(16)	0.087(13)	0.149(9)
$\mathcal{G}_N^1$	1	0.058(18)	0.076(9)	0.085(8)	0.150(5)
S_N, S_N'	1	0.074(20)	0.097(10)	0.106(7)	0.167(5)
$\mathcal{G}_N^2$	1	0.058(17)	0.074(9)	0.083(8)	0.146(5)
$\mathcal{H}_N$	3	0.052(66)	0.066(33)	0.074(26)	0.112(16)
$\mathcal{G}_N^1$	3	0.052(35)	0.066(18)	0.075(13)	0.116(8)
S_N, S_N'	3	0.050(28)	0.062(14)	0.068(11)	0.099(7)
$\mathcal{G}_N^2$	3	0.050(33)	0.065(17)	0.073(13)	0.103(8)
$\mathcal{H}_N$	6	0.058(140)	0.068(62)	0.074(45)	0.100(27)
$\mathcal{G}_N^1$	6	0.058(68)	0.068(32)	0.075(23)	0.099(15)
S_N, S_N'	6	0.047(40)	0.055(21)	0.059(17)	0.078(9)
$\mathcal{G}_N^2$	6	0.050(65)	0.062(33)	0.069(23)	0.099(15)

Table 2. $\left\|\widetilde{v}-\widetilde{v}_l^\delta\right\| / \|\widetilde{v}\|$ of CGNE regularization for the sphere.

Method	$k\backslash\delta$	$0.001(l)$	$0.005(l)$	$0.01(l)$	$0.05(l)$
$\mathcal{H}_N$	1	0.182(1288)	0.185(1195)	0.190(1195)	0.238(1132)
$\mathcal{G}_N^1$	1	0.183(262)	0.187(170)	0.192(170)	0.243(106)
S_N, S'_N	1	0.060(352)	0.082(236)	0.101(186)	0.188(107)
$\mathcal{G}_N^2$	1	0.190(588)	0.215(343)	0.234(254)	0.299(156)
$\mathcal{H}_N$	3	0.134(1701)	0.160(1382)	0.171(1330)	0.275(1263)
$\mathcal{G}_N^1$	3	0.134(675)	0.161(356)	0.173(304)	0.278(241)
S_N, S'_N	3	0.050(422)	0.058(281)	0.065(225)	0.134(178)
$\mathcal{G}_N^2$	3	0.200(839)	0.249(552)	0.263(385)	0.337(276)
$\mathcal{H}_N$	6	0.165(1840)	0.224(1834)	0.276(1679)	0.425(1295)
$\mathcal{G}_N^1$	6	0.166(814)	0.224(808)	0.277(652)	0.427(270)
S_N, S'_N	6	0.090(789)	0.147(787)	0.203(548)	0.359(348)
$\mathcal{G}_N^2$	6	0.189(866)	0.253(819)	0.331(689)	0.470(319)

Table 3. $\left\|\widetilde{v}-\widetilde{v}_l^\delta\right\| / \|\widetilde{v}\|$ of TSVD regularization for the fuselage.

Method	$k\backslash\delta$	$0.001(l)$	$0.005(l)$	$0.01(l)$	$0.05(l)$
$\mathcal{H}_N$	1	0.152(33)	0.15572(28)	0.162(25)	0.218(14)
$\mathcal{G}_N^1$	1	0.156(32)	0.160(30)	0.167(25)	0.220(18)
S_N, S'_N	1	0.054(43)	0.070(25)	0.089(22)	0.183(13)
$\mathcal{G}_N^2$	1	0.185(150)	0.208(65)	0.223(47)	0.289(26)
$\mathcal{H}_N$	3	0.130(150)	0.139(78)	0.153(70)	0.276(39)
$\mathcal{G}_N^1$	3	0.131(150)	0.141(93)	0.155(82)	0.281(49)
S_N, S'_N	3	0.047(53)	0.054(40)	0.066(34)	0.177(22)
$\mathcal{G}_N^2$	3	0.240(150)	0.244(150)	0.256(124)	0.361(64)
$\mathcal{H}_N$	6	0.292(150)	0.297(150)	0.309(150)	0.405(57)
$\mathcal{G}_N^1$	6	0.331(150)	0.332(150)	0.336(150)	0.408(75)
S_N, S'_N	6	0.122(150)	0.153(146)	0.197(70)	0.346(25)
$\mathcal{G}_N^2$	6	0.421(150)	0.421(150)	0.421(150)	0.451(114)

Table 4. $\left\|\widetilde{v}-\widetilde{v}_l^\delta\right\| / \|\widetilde{v}\|$ of CGNE regularization for the fuselage.