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Schwarz–Christoffel Mapping of the Annulus*

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Dedicated to the memory of Peter Henrici

Abstract. A new derivation of the Schwarz–Christoffel (S–C) transformation for doubly connected domains is given. The work is based on constructing the pre-Schwarzian by reflection of singularities. A derivation of the simply connected S–C map without appeal to Liouville's theorem is a byproduct of this work.

Key words. conformal mapping, Schwarz–Christoffel transformation

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1. Introduction.

The Schwarz–Christoffel (S–C) formula [He86]

$$f(z) = A\int^{z}\prod_{k=1}^{m}\left[\Theta\left(\frac{\zeta}{\mu z_{0,k}}\right)\right]^{-\beta_{0,k}}\prod_{k=1}^{n}\left[\Theta\left(\frac{\mu\zeta}{z_{1,k}}\right)\right]^{\beta_{1,k}}d\zeta + B \tag{1}$$

for the conformal map of the annulus $\mu < |z| < 1$ onto a doubly connected polygonal domain is well known [Kom45], [vKS59], [DT]. The prevertices, $z_{\nu,k}$, and the turning parameters, $\beta_{\nu,k}$, are defined later. Motivated by the derivation in [Ne52] of the S–C mapping formula for the half-plane and the disk onto a simply connected polygonal domain, we give a new derivation of the annulus formula that is based on the construction of a global singularity function $S(z)$ for the analytic continuation of the pre-Schwarzian, $f''(z)/f'(z)$, of the mapping function f. One new and especially interesting feature of our derivation is the construction of the singularity function by repeated reflections that generate the infinite products giving the function Θ in (1). The proof of (1) in [He86] is based on the remarkable appearance of the function

$$\Theta(w) = \prod_{\nu=0}^{\infty}\left(1-\mu^{2\nu+1}w\right)\left(1-\mu^{2\nu+1}/w\right) \tag{2}$$

and clever use of the argument principle. Elliptic functions enter Komatu's work [Kom45] through his use of Villat's integral representation for functions analytic in

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the annulus. However, unlike proofs that depend upon a priori knowledge and use of elliptic functions, in our approach the function Θ is generated in a natural and unavoidable manner. Furthermore, our constructive derivation is natural from the standpoint of numerical implementation of the mapping procedure. The infinite sums must be truncated, and our construction points toward efficient ways of doing this.

A second basic feature of our work is the proof of the identity $f''(z)/f'(z) = S(z)$ based on the maximum principle for harmonic functions, which also gives a new variant of the proof of the S–C formula for the simply connected case. It can be argued that the derivation given in [He86] using the argument principle is more compact. On the other hand, our proof uses only the construction given of our singularity function and does not require properties of theta functions. In fact, a derivation of these functions is implicit in our construction. The integral formulas for f are the starting point for recent implementations of efficient numerical methods for computing S–C mappings [D96], [Da88], [Hu98].

2. S–C for the Unit Disk. In this section we sketch a derivation of the S–C mapping formula for simply connected polygonal regions that will aid the reader's understanding of our work in section 3. Let $\mathbb{D}$ denote the unit disk and $f : \mathbb{D} \longmapsto \mathbb{P}$ a conformal mapping onto a polygonal domain $\mathbb{P}$ with m vertices $w_1, \ldots, w_m$, interior angles $\alpha_k\pi$, and turning angles $\beta_k\pi$, $\beta_k = 1 - \alpha_k$, at each vertex and *prevertices* $z_k = f^{-1}(w_k) = e^{i\theta_k}$ on the unit circle. The mapping extends continuously to a $1-1$ mapping of $cl(\mathbb{D})$ onto $cl(\mathbb{P})$ and extends analytically across each of the m arcs $\gamma_k = \text{arc}(z_k, z_{k+1}) = \{e^{it} : \theta_k < t < \theta_{k+1}\}$ of the unit circle between the prevertices. The S–C mapping formula is established by proving that

$$\frac{f''(z)}{f'(z)} = -\sum_{k=1}^{m} \frac{\beta_k}{z - z_k}, \qquad z \in \mathbb{D}. \tag{3}$$

Integrations then yield the familiar S–C mapping formula

$$f(z) = A \int_0^z \frac{1}{\prod_{k=1}^{m} (\zeta - z_k)^{\beta_k}} d\zeta + B.$$

Recall that elementary geometry of the turning tangent shows that $-1 \le \beta_k < 1$ and $\sum_{k=1}^{m} \beta_k = 2$.

As in [Ne52] we begin the proof of (3) by considering the analytic continuation of f to $\Delta = \{\, |z| > 1\}$ by reflection across an arc γ_k. Such an extension, $\widetilde{f}_k$, has the form $\widetilde{f}_k(z) = f(z),\ z \in \mathbb{D} \cup \gamma_k,\ \ \widetilde{f}_k(z) = a_k \overline{f(1/\overline{z})} + b_k,\ z \in \Delta,$ with a_k, b_k determined by the line containing the edge $f(\gamma_k)$ joining w_k and w_{k+1} in the boundary of $\mathbb{P}$. This reflection maps Δ conformally onto $\mathbb{P}_k$, the polygon obtained by reflecting across $f(\gamma_k)$. If one repeats the continuation process continuing $\widetilde{f}_k$ by reflection across an arc γ_j and the corresponding side of $\mathbb{P}_k$, the resulting extension $\widetilde{f}_{k,j}$ differs from the original function element $f(z)$ in $\mathbb{D}$. However, the values $\widetilde{f}_{k,j}(z)$ for $z \in \mathbb{D} \cup \gamma_k \cup \gamma_j$ are obtained from those of f by the composition of two reflections across lines, and therefore $\widetilde{f}_{k,j}(z) = Af(z) + B,\ z \in \mathbb{D} \cup \gamma_k \cup \gamma_j$, for some $A, B \in \mathbb{C}$ that depend on the particular pair of reflections.

By repeated application of the reflection process through the arcs γ_k, one obtains from the initial function element $(f, \mathbb{D})$ a global (many-valued) analytic function $\widehat{f}$ defined on the punctured plane $\mathbb{C}\backslash\{z_1, \ldots, z_m\}$. Any two values, $\widehat{f}_r\,(z)$ and $\widehat{f}_s\,(z)$,

of $\widehat{f}$ at a point $z \in \mathbb{C} \setminus \{z_1, \ldots, z_m\}$ are related by the composition of an even number of reflections in lines, and hence $\widehat{f}_s(z) = \alpha \widehat{f}_r(z) + \beta$ for some $\alpha, \beta \in \mathbb{C}$. The pre-Schwarzian of f, $f''(z)/f'(z)$, is invariant under affine maps $w \longmapsto aw + b$, i.e., $(af(z)+b)''/(af(z)+b)' = f''(z)/f'(z)$. Thus, if one begins with the pre-Schwarzian of the mapping in $\mathbb{D}$, the reflection process yielding the many-valued $\widehat{f}$ also defines a global analytic pre-Schwarzian, $\widehat{f}''(z)/\widehat{f}'(z)$, that is defined and *single-valued* on $\mathbb{C}\backslash\{z_1, \ldots, z_m\}$. By familiar reasoning [Ne52], $(f(z) - w_k)^{1/\alpha_k}$ is analytic in a neighborhood of z_k and therefore has a local expansion $(f(z) - w_k)^{1/\alpha_k} = (z - z_k)\hat{h}_k(z)$, where $\hat{h}_k(z)$ is analytic and nonvanishing near z_k. This gives the local expansion $f''(z)/f'(z) = \frac{-\beta_k}{z - z_k} + H_k(z)$, where $H_k(z)$ is analytic in a neighborhood of z_k. That is, the singularities of $\widehat{f}''(z)/\widehat{f}'(z)$ at the prevertices $z_1, \ldots, z_m$ are simple poles with residues $-\beta_1, \ldots, -\beta_m$, respectively. Thus

$$\frac{\widehat{f}''(z)}{\widehat{f}'(z)} - S(z) = \frac{\widehat{f}''(z)}{\widehat{f}'(z)} - \sum_{k=1}^{m} \frac{-\beta_k}{z - z_k}$$

is an entire function. Here $S(z) = \sum_{k=1}^{m} \frac{-\beta_k}{z - z_k}$ and, in general, we shall denote the "singularity function"of a globally defined pre-Schwarzian by $S(z)$.

We now complete the derivation by showing that $z[\widehat{f}''(z)/\widehat{f}'(z) - S(z)]$ is identically zero. The tangent angle $\psi(t) = \arg\{ie^{it} f'(e^{it})\}$ of the boundary of $\mathbb{P}$ is constant on each of the arcs γ_k between the prevertices on the unit circle, and hence

$$\psi'(t) = 1 + \mathrm{Re}\left\{ e^{it} \frac{f''(e^{it})}{f'(e^{it})} \right\} = 0, \qquad t \in [0, 2\pi]\backslash\{t_1, \ldots, t_m\}.$$

The Moebius transformation $w = \zeta/(1-\zeta)$ maps $|\zeta| = 1, \zeta \neq 1$ onto the line $\mathrm{Re}\{w\} = -1/2$. Hence for $z = e^{it}$,

$$\mathrm{Re}\{e^{it} S(e^{it})\} = \sum_{k=1}^{m} \beta_k \mathrm{Re}\left\{ \frac{z\overline{z}_k}{1 - z\overline{z}_k} \right\} = \sum_{k=1}^{m} \beta_k(-1/2) = -1,$$

$t \in [0, 2\pi]\backslash\{t_1, \ldots, t_m\}$. By continuity the harmonic function $\mathrm{Re}\{zf''(z)/f'(z) - zS(z)\}$ is identically zero on the unit circle and hence throughout the closed unit disk. Consequently the analytic function $zf''(z)/f'(z) - zS(z)$ is constant, and the constant is zero since that is the value at the origin. This establishes the formula (3).

We note that the argument has the following steps (not given in this order above):

(a) $\mathrm{Re}\{zf''(z)/f'(z)\} = -1$ on $|z| = 1, z \neq z_k$.

(b) $\mathrm{Re}\{zS(z)\} = -1$ at the same points.

(c) $\mathrm{Re}\{zf''(z)/f'(z) - zS(z)\} = 0$ on $|z| \leq 1$ by continuity and the maximum principle.

(d) $zf''(z)/f'(z) - zS(z) = i\lambda$, and $\lambda = 0$ since the left-hand side is 0.

Of course, the key is (b) since the correct construction of $S(z)$ leads to this condition.

3. S–C for an Annulus. We let C_ρ denote the circle $C_\rho = \{z : |z| = \rho\}$ and $\mathcal{A}_{r,R} = \{r < |z| < R\}$, the annulus. Also, $\overline{a, b}$ will denote the line segment joining two points $a, b \in \mathbb{C}$.

We now derive the formula for the conformal mapping, f, of the annulus $\mathcal{A}_{\mu,1}$ onto a bounded, doubly connected, polygonal region $\mathbb{P}$ with modulus $1/\mu$, outer polygonal boundary component Γ_0, and inner polygonal boundary component Γ_1 contained in

the interior of Γ_0. Both Γ_0 and Γ_1 are given the positive (counterclockwise) orientation. Let the m vertices of Γ_0 and the n vertices of Γ_1 be denoted $w_{0,j},\ j=1,\dots,m$, and $w_{1,k},\ k=1,\dots,n$, respectively. The vertex angles of the Γ_ν at the corners $w_{\nu,j}$, measured from the interior of $\mathbb{P}$, are denoted by $\pi\alpha_{\nu,j}$, where $0<\alpha_{\nu,j}\le 2$. From elementary considerations one has the relations

$$\sum_{j=1}^{m}\alpha_{0,j}=m-2, \qquad \sum_{j=1}^{n}\alpha_{1,j}=n+2.$$

As a point traverses the outer boundary, Γ_0, the turning of the tangent at a vertex, $w_{0,j}$, is $\pi(1-\alpha_{0,j})$, but for a point traversing the inner boundary, Γ_1, the turning at a vertex, $w_{1,j}$, is $\pi(\alpha_{1,j}-1)$ (since $\mathbb{P}$ now lies to the "right"). We let $\beta_{0,j}=(1-\alpha_{0,j})$ and $\beta_{1,j}=(\alpha_{1,j}-1)$ denote the *turning parameters* and note the important relations

$$\sum_{j=1}^{m}\beta_{0,j}=2=\sum_{j=1}^{n}\beta_{1,j}. \tag{4}$$

The conformal map $f:\mathcal{A}_{\mu,1}\longmapsto\mathbb{P}$ extends continuously to a homeomorphism of $cl\,(\mathcal{A}_{\mu,1})$ onto $cl\,(\mathbb{P})$ and is chosen so that $f(C_\mu)=\Gamma_1$ and $f(C_1)=\Gamma_0$.

We begin with a cursory overview of our derivation and then follow with the details. The critical idea is to first consider the global analytic (multiple-valued) function $\widehat{f}$ obtained by unlimited analytic continuation (via reflections) of the initial Laurent function element $(f,\mathcal{A}_{\mu,1})$ to $\mathbb{C}\backslash V$, where V is the countably infinite union of the original $m+n$ prevertices, $z_{\nu,k}=f^{-1}(w_{\nu,k})$, and all of their reflections. The reflection process covers $0<|z|<\infty$ with concentric annuli and their boundary circles. The multiple-valued global analytic function $\widehat{f}$ has the property that any two values, $\widehat{f}_r(z)$ and $\widehat{f}_s(z)$, of $\widehat{f}$ at a point $z\in\mathbb{C}\backslash V$ are related by an even number of reflections across straight lines, and hence $\widehat{f}_s(z)=a\widehat{f}_r(z)+b$ for some $a,b\in\mathbb{C}$. One passes to the pre-Schwarzian $f''(z)/f'(z)$, uses its affine invariance, and obtains a global pre-Schwarzian $\widehat{f}''(z)/\widehat{f}'(z)$ that is defined, analytic, and single-valued on $\mathbb{C}\backslash V$. In the remaining steps of the derivation, one first identifies and analyzes the singularities of the pre-Schwarzian at the points in V and constructs its singularity function, $S(z)$. Finally, one argues that $\widehat{f}''(z)/\widehat{f}'(z)=S(z)$ and then obtains the formula for $f(z)$ by integrations in the original annulus, $\mathcal{A}_{\mu,1}$.

Proceeding with the details, we first "track" the reflections of the prevertices. Recall that the reflection of a point z in a circle C_R is $z^*=R^2/\overline{z}=R^2z/\left|z\right|^2$. When $\mathcal{A}_{\mu,1}$ is reflected inward (toward 0) to its reflected image $\mathcal{A}_{\mu^2,\mu}$, the prevertices $\{z_{0,k}\}_1^m$ on C_1 reflect to the set of points $\left\{\mu^2 z_{0,k}\right\}_1^m$ and the prevertices $\{z_{1,k}\}_1^n$ on C_μ stay fixed. These points, $\left\{\mu^2 z_{0,k}\right\}_1^m$ and $\{z_{1,k}\}_1^n$, are prevertices of a conformal map of $\mathcal{A}_{\mu^2,\mu}$ onto a doubly connected polygonal region that is the reflection of $\mathbb{P}$ across an "inner" side $\overline{w_{1,k},w_{1,k+1}}$ of its polygonal boundary Γ_1. The conformal map of $\mathcal{A}_{\mu^2,\mu}$ is obtained from f by reflection across $\mathrm{arc}(z_{1,k},z_{1,k+1})=f^{-1}(\overline{w_{1,k},w_{1,k+1}})$. Clearly, there are n such extensions, one for each arc of C_μ and corresponding inner side of $\mathbb{P}$. Each pair of these extensions are related by two reflections and hence differ by an affine map $w\longmapsto aw+b$ (a and b depend on the pair of extensions).

The reflection/extension of f continues inward by reflecting $\mathcal{A}_{\mu^2,\mu}$ across C_{μ^2} to an annulus $\mathcal{A}_{\mu^3,\mu^2}$. The prevertices on the boundary of $\mathcal{A}_{\mu^3,\mu^2}$ are the unchanged set $\left\{\mu^2 z_{0,k}\right\}_1^m$ and $\left\{\mu^2 z_{1,k}\right\}_1^n$, the reflections in C_{μ^2} of $\{z_{1,k}\}_1^n$. Each branch of the extension of $\widehat{f}$ to $\mathcal{A}_{\mu^3,\mu^2}$ comes from a reflection across an arc $\mu^2\gamma_k=\mathrm{arc}\left(\mu^2 z_{0,k},\mu^2 z_{0,k+1}\right)$

of C_{μ^2} and across its corresponding "outer" boundary edge $\widehat{f}\left(\mu^2\gamma_k\right)$ of one of the first sets of inward reflections of $\mathbb{P}$. In general, one continues the inward reflection/extension process across the prevertex-free boundary arcs of $\mathcal{A}_{\mu^{s+1},\mu^s}$ to $\mathcal{A}_{\mu^{s+2},\mu^{s+1}}$, $s = 0, 1, 2, \ldots$, to generate the complete, many-valued extension $\widehat{f}$ of f to the punctured unit disk, $\{0 < |z| < 1\}$, with singularities at the prevertex points

$$\bigcup_{N=0}^{\infty}\left[\left\{\mu^{2N}z_{0,k}\right\}_{k=1}^{m}\cup\left\{\mu^{2N}z_{1,k}\right\}_{k=1}^{n}\right]. \tag{5}$$

In the same manner, one reflects $\mathcal{A}_{\mu,1}$ outward (toward ∞) across C_1 to the annulus $\mathcal{A}_{1,1/\mu}$ and, in general, $\mathcal{A}_{\mu^{-s},\mu^{-(s+1)}}$ to $\mathcal{A}_{\mu^{-(s+1)},\mu^{-(s+2)}}$ in order to generate the complete extension of $\widehat{f}$ to the exterior of the unit disk with singularities at the prevertex points

$$\bigcup_{N=1}^{\infty}\left[\left\{\mu^{-2N}z_{1,k}\right\}_{k=1}^{n}\cup\left\{\mu^{-2N}z_{0,k}\right\}_{k=1}^{m}\right]. \tag{6}$$

The reflection/extension process just described, beginning with the initial element $(f, \mathcal{A}_{\mu,1})$, generates the (many-valued) global analytic function $\widehat{f}$ defined on $\mathbb{C}\backslash V\backslash\{0\}$, where V is the union of the sets (5) and (6) of prevertices. However, the many-valued $\widehat{f}$ defines a global analytic pre-Schwarzian $\widehat{f}''(z)/\widehat{f}'(z)$ that is defined and single-valued on $\mathbb{C}\backslash V\backslash\{0\}$. Indeed, it is easy to show that any two values of $\widehat{f}(z)$ at a point $z \in \mathbb{C}\backslash V\backslash\{0\}$ are related by an even number of reflections in straight lines and hence by an affine map $w \longmapsto aw + b$. The single-valuedness of $\widehat{f}''(z)/\widehat{f}'(z)$ then follows from its affine invariance.

The next step is to analyze the singularities of $\widehat{f}''(z)/\widehat{f}'(z)$ at the prevertices V and construct $S(z)$, the singularity function of the global pre-Schwarzian. As in section 2, local analysis of $(f(z) - f(z_k))^{1/\alpha_k}$ near a prevertex shows that the singularities of $\widehat{f}''(z)/\widehat{f}'(z)$ at the "outer edge" prevertices $\bigcup_{N\in\mathbb{Z}}\{\mu^{2N}z_{0,k}\}_{k=1}^{m}$ are simple poles with residues $-\beta_{0,k} = (\alpha_{0,k} - 1), 1 \le k \le m$, and that the singularities at the "inner edge" prevertices $\bigcup_{N\in\mathbb{Z}}\left\{\mu^{2N}z_{1,k}\right\}_{k=1}^{n}$ are simple poles with residues $\beta_{1,k} = (\alpha_{1,k} - 1), 1 \le k \le n$.

We want $S(z)$ to have the property that $\widehat{f}''(z)/\widehat{f}'(z) - S(z)$ is analytic in $0 < |z| < \infty$. Thus it appears that we should form

$$\begin{aligned} S(z) = &\sum_{k=1}^{m} -\beta_{0,k}\left\{\frac{1}{z - z_{0,k}} + \frac{1}{z - \mu^2 z_{0,k}} + \frac{1}{z - \mu^{-2}z_{0,k}} + \cdots\right\} \\ &+ \sum_{k=1}^{n}\beta_{1,k}\left\{\frac{1}{z - z_{1,k}} + \frac{1}{z - \mu^{-2}z_{1,k}} + \frac{1}{z - \mu^2 z_{1,k}} + \cdots\right\}, \end{aligned}$$

but this ignores the obvious question of convergence of the series. We therefore begin by arranging the reflections and the corresponding singularities of the global pre-Schwarzian in groups of four, as follows:

$$\begin{aligned} S_N(z) = &\sum_{k=1}^{m}\left\{-\beta_{0,k}\sum_{\nu=0}^{N}\left(\frac{1}{z - z_{0,k}\mu^{-2\nu}} + \frac{1}{z - z_{0,k}\mu^{2(\nu+1)}}\right)\right\} \\ &+ \sum_{k=1}^{n}\left\{\beta_{1,k}\sum_{\nu=0}^{N}\left(\frac{1}{z - z_{1,k}\mu^{2\nu}} + \frac{1}{z - z_{1,k}\mu^{-2(\nu+1)}}\right)\right\}, \end{aligned}$$

and defining $S(z) = \lim_{N\to\infty} S_N(z)$. It is clear that $\widehat{f}''(z)/\widehat{f}'(z) - S_N(z)$ is analytic in the annulus $\mu^{2N+3} < |z| < \mu^{-2(N+1)}$, but convergence as $N \to \infty$ is not clear. The terms involving $z_{0,k}\mu^{2(\nu+1)}$ and those with $z_{1,k}\mu^{2\nu}$ appear to prevent convergence since they tend to $1/z$ as $\nu \to \infty$. However,

$$\frac{1}{z - z_{0,k}\mu^{2(\nu+1)}} = \frac{1}{z} + \frac{\mu^{2(\nu+1)}\left(z_{0,k}/z^2\right)}{1 - \mu^{2(\nu+1)}\left(z_{0,k}/z\right)}$$

and

$$\frac{1}{z - z_{1,k}\mu^{2\nu}} = \frac{1}{z} + \frac{\mu^{2\nu}\left(z_{1,k}/z^2\right)}{1 - \mu^{2\nu}\left(z_{1,k}/z\right)},$$

and since (4) holds, the $1/z$ terms in $S_N(z)$ add out to yield

$$\begin{aligned} S_N(z) = &-\sum_{k=1}^{m}\left\{\beta_{0,k}\sum_{\nu=0}^{N}\left(\frac{1}{z - z_{0,k}\mu^{-2\nu}} + \frac{\mu^{2(\nu+1)}\left(z_{0,k}/z^2\right)}{1 - \mu^{2(\nu+1)}\left(z_{0,k}/z\right)}\right)\right\} \\ &+\sum_{k=1}^{n}\left\{\beta_{1,k}\sum_{\nu=0}^{N}\left(\frac{\mu^{2\nu}\left(z_{1,k}/z^2\right)}{1 - \mu^{2\nu}\left(z_{1,k}/z\right)} + \frac{1}{z - z_{1,k}\mu^{-2(\nu+1)}}\right)\right\}. \end{aligned}$$

Finally, since

$$\begin{aligned} S_N(z) = &-\sum_{k=1}^{m}\left\{\beta_{0,k}\sum_{\nu=0}^{N}\left(\frac{-\mu^{2\nu}/z_{0,k}}{1 - \mu^{2\nu}\left(z/z_{0,k}\right)} + \frac{\mu^{2(\nu+1)}\left(z_{0,k}/z^2\right)}{1 - \mu^{2(\nu+1)}\left(z_{0,k}/z\right)}\right)\right\} \\ &+\sum_{k=1}^{n}\left\{\beta_{1,k}\sum_{\nu=0}^{N}\left(\frac{\mu^{2\nu}\left(z_{1,k}/z^2\right)}{1 - \mu^{2\nu}\left(z_{1,k}/z\right)} - \frac{\mu^{2(\nu+1)}/z_{1,k}}{1 - \mu^{2(\nu+1)}\left(z/z_{1,k}\right)}\right)\right\}, \end{aligned} \tag{7}$$

it is easily seen that the sequence $\{S_N(z)\}$ converges uniformly on compact subsets of $\mathbb{C}\backslash V\backslash\{0\}$ and hence defines there the singularity function $S(z) = \lim_{N\to\infty} S_N(z)$, i.e.,

$$\begin{aligned} S(z) = &-\sum_{\nu=0}^{\infty}\sum_{k=1}^{m}\beta_{0,k}\left(\frac{-\mu^{2\nu}/z_{0,k}}{1 - \mu^{2\nu}\left(z/z_{0,k}\right)} + \frac{\mu^{2(\nu+1)}\left(z_{0,k}/z^2\right)}{1 - \mu^{2(\nu+1)}\left(z_{0,k}/z\right)}\right) \\ &+\sum_{\nu=0}^{\infty}\sum_{k=1}^{n}\beta_{1,k}\left(\frac{\mu^{2\nu}\left(z_{1,k}/z^2\right)}{1 - \mu^{2\nu}\left(z_{1,k}/z\right)} - \frac{\mu^{2(\nu+1)}/z_{1,k}}{1 - \mu^{2(\nu+1)}\left(z/z_{1,k}\right)}\right). \end{aligned} \tag{8}$$

The construction of $S(z)$ ensures that $\widehat{f}''(z)/\widehat{f}'(z) - S(z)$ is analytic in $0 < |z| < \infty$, and we claim that, in fact, $\widehat{f}''(z)/\widehat{f}'(z) - S(z) = 0$ in $0 < |z| < \infty$. We prove our claim as follows:

(a) Observe that $\operatorname{Re}\{zf''(z)/f'(z)\} = -1$ on each of the boundary circles $|z| = \mu$ and $|z| = 1$ except at the prevertices.

(b) Prove that $\operatorname{Re}\{zS(z)\} = -1$ on the same boundary arcs.

(c) Deduce that $\operatorname{Re}\{zf''(z)/f'(z) - zS(z)\} = 0$ on the full boundary of $\mathcal{A}_{\mu,1}$ and hence throughout the annulus by the maximum principle for harmonic functions.

(d) From the previous step we then have $zf''(z)/f'(z) - zS(z) = i\lambda$, λ a real constant. With an argument involving Laurent series we show that $\lambda = 0$ and hence the truth of our claim.

Proof of (a). The tangent angle $\psi(t) = \arg\{ire^{it}f'(re^{it})\}$ of the boundary $\mathcal{A}_{\mu,1}$, $r = \mu$ or 1, is constant on each of the arcs between prevertices. Hence $\psi'(t) = \mathrm{Re}\{z\frac{f''(z)}{f'(z)} + 1\} = 0, |z| = \mu, 1, z \neq z_{\nu,k}$, and this proves (a).

Proof of (b). First show that $\mathrm{Re}\,\{zS_N(z)\}_{|z|=1} = -1 + O\left(\mu^{2(N+1)}\right)$ on the arcs of $|z| = 1$ between prevertices, which clearly implies $\mathrm{Re}\,\{zS(z)\}_{|z|=1} = -1$, $z \neq z_{0,k}$. We begin with the terms inside the $\beta_{0,k}$ sum in formula (7) and see that

$$
\begin{aligned}
(9)\quad & -\mathrm{Re}\left\{ z\sum_{\nu=0}^{N}\left(\frac{-\mu^{2\nu}/z_{0,k}}{1-\mu^{2\nu}\,(z/z_{0,k})} + \frac{\mu^{2(\nu+1)}\left(z_{0,k}/z^2\right)}{1-\mu^{2(\nu+1)}\,(z_{0,k}/z)}\right)\right\}_{|z|=1} \\
&= \mathrm{Re}\left\{\sum_{\nu=0}^{N}\left(\frac{\mu^{2\nu}(z/z_{0,k})}{1-\mu^{2\nu}\,(z/z_{0,k})} - \frac{\mu^{2(\nu+1)}\,(z_{0,k}/z)}{1-\mu^{2(\nu+1)}\,(z_{0,k}/z)}\right)\right\}_{|z|=1} \\
&= \mathrm{Re}\left\{\sum_{\nu=0}^{N}\left(\frac{\mu^{2\nu}(z/z_{0,k})}{1-\mu^{2\nu}\,(z/z_{0,k})} - \frac{\mu^{2(\nu+1)}\overline{(z/z_{0,k})}}{1-\mu^{2(\nu+1)}\overline{(z/z_{0,k})}}\right)\right\}_{|z|=1}, \text{ (since } |z/z_{0,k}| = 1), \\
&= \mathrm{Re}\frac{z/z_{0,k}}{1-z/z_{0,k}} + \sum_{\nu=1}^{N}\mathrm{Re}\left(\frac{\mu^{2\nu}(z/z_{0,k})}{1-\mu^{2\nu}\,(z/z_{0,k})} - \frac{\mu^{2\nu}\overline{(z/z_{0,k})}}{1-\mu^{2\nu}\overline{(z/z_{0,k})}}\right)_{|z|=1} \\
&\quad -\mu^{2(N+1)}\mathrm{Re}\frac{z/z_{0,k}}{1-\mu^{2(N+1)}z/z_{0,k}} \\
&= -\frac{1}{2} + O\left(\mu^{2(N+1)}\right).
\end{aligned}
$$

Turning to the terms inside the $\beta_{1,k}$ sum in formula (7) and multiplying by z with $|z| = 1$ we find that

$$
\begin{aligned}
&\mathrm{Re}\sum_{\nu=0}^{N} z\left(\frac{\mu^{2\nu}\left(z_{1,k}/z^2\right)}{1-\mu^{2\nu}\,(z_{1,k}/z)} - \frac{\mu^{2(\nu+1)}/z_{1,k}}{1-\mu^{2(\nu+1)}\,(z/z_{1,k})}\right) \\
&= \mathrm{Re}\sum_{\nu=0}^{N}\left(\frac{\mu^{2\nu}\,(z_{1,k}/z)}{1-\mu^{2\nu}\,(z_{1,k}/z)} - \frac{\mu^{2(\nu+1)}\,(z/z_{1,k})}{1-\mu^{2(\nu+1)}\,(z/z_{1,k})}\right) \\
&= \mathrm{Re}\sum_{\nu=0}^{N}\left(\frac{\mu^{2\nu}\,(z_{1,k}/z)}{1-\mu^{2\nu}\,(z_{1,k}/z)} - \frac{\mu^{2\nu}\overline{(z_{1,k}/z)}}{1-\mu^{2\nu}\overline{(z_{1,k}/z)}}\right) = 0,
\end{aligned}
$$

where we have used the fact that $|z| = 1$ and $|z_{1,k}| = \mu$ imply that $z/z_{1,k} = \mu^{-2}\overline{(z_{1,k}/z)}$. Summing (9) over $\sum_{k=1}^{m}\beta_{0,k}$ yields the result $\mathrm{Re}\,\{zS_N(z)\}_{|z|=1} = -1 + O\left(\mu^{2(N+1)}\right)$, $z \neq z_{0,k}$. In a similar manner we note that on the circle $|z| = \mu$, $z_{1,k}/z = \overline{(z_{1,k}/z)}$, $z_{0,k}/z = \mu^{-2}\overline{(z/z_{0,k})}$, and hence we argue that on arcs of $|z| = \mu$ between prevertices, one has $\mathrm{Re}\,\{zS_N(z)\}_{|z|=\mu} = -1 + O\left(\mu^{2(N+1)}\right)$. This proves (b).

Proof of (c). From (a) and (b) it now follows that $\mathrm{Re}\,\{zf''(z)/f'(z) - zS(z)\} = 0$ on the full boundary of $\mathcal{A}_{\mu,1}$ since the prevertex singularities cancel, and hence throughout the annulus by the maximum principle.

Proof of (d). From (c) we have $zf''(z)/f'(z) - zS(z) = i\lambda$, λ a real constant. Let $zf''(z)/f'(z) = \sum_{\nu=-\infty}^{\infty} c_\nu z^\nu$ and $zS(z) = \sum_{\nu=-\infty}^{\infty} s_\nu z^\nu$ for $z \in \mathcal{A}_{\mu,1}$. We now show that $c_0 = s_0 = 0$, and therefore $i\lambda = c_0 - s_0 = 0$. Fixing r, $\mu < r < 1$, we have the

tangent angle $\psi'(t) = \arg\{ire^{it}f'(re^{it})\}$ in the image domain with

$$2\pi = \int_0^{2\pi} \psi'(t)dt = \int_0^{2\pi} 1 + \mathrm{Re}\{re^{it}f''(re^{it})/f'(re^{it})\}dt.$$

Thus $\int_0^{2\pi} \mathrm{Re}\{re^{it}f''(re^{it})/f'(re^{it})\}dt = 0$. Furthermore,

$$\int_0^{2\pi} \mathrm{Im}\{re^{it}f''(re^{it})/f'(re^{it})\}dt = 0$$

since $\log|f'(re^{it})|$ is single-valued. This shows that

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} re^{it}f''(re^{it})/f'(re^{it})dt = 0.$$

The uniform convergence on $|z| = r$ gives

$$s_0 = \frac{1}{2\pi i}\int_{|z|=r} S(z)dz = \lim_{N\to\infty}\frac{1}{2\pi i}\int_{|z|=r} S_N(z)dz.$$

Using the first form of $S_N(z)$ above and the residue theorem, it is easy to show that

$$\frac{1}{2\pi i}\int_{|z|=r} S_N(z)dz = \sum_{k=1}^{n}\beta_{1,k}(N+1) - \sum_{k=1}^{m}\beta_{0,k}(N+1) = 0$$

and hence that $s_0 = 0$.

Now, for $z \in \mathcal{A}_{\mu,1}$, we have the formula

$$\begin{aligned}
\frac{f''(z)}{f'(z)} &= -\sum_{k=1}^{m}\beta_{0,k}\sum_{\nu=0}^{\infty}\left(\frac{-\mu^{2\nu}/z_{0,k}}{1-\mu^{2\nu}\left(z/z_{0,k}\right)} + \frac{\mu^{2(\nu+1)}\left(z_{0,k}/z^2\right)}{1-\mu^{2(\nu+1)}\left(z_{0,k}/z\right)}\right)\\
&\quad + \sum_{k=1}^{n}\beta_{1,k}\sum_{\nu=0}^{\infty}\left(\frac{\mu^{2\nu}\left(z_{1,k}/z^2\right)}{1-\mu^{2\nu}\left(z_{1,k}/z\right)} - \frac{\mu^{2(\nu+1)}/z_{1,k}}{1-\mu^{2(\nu+1)}\left(z/z_{1,k}\right)}\right)\\
&= \sum_{k=1}^{m} -\beta_{0,k}\sum_{\nu=0}^{\infty}\frac{d}{dz}\log\left\{\left(1-\mu^{2\nu}z/z_{0,k}\right)\left(1-\mu^{2(\nu+1)}z_{0,k}/z\right)\right\}\\
&\quad + \sum_{k=1}^{n}\beta_{1,k}\sum_{\nu=0}^{\infty}\frac{d}{dz}\log\left\{\left(1-\mu^{2(\nu+1)}z/z_{1,k}\right)\left(1-\mu^{2\nu}z_{1,k}/z\right)\right\}
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{d}{dz}\log f'(z) &= \sum_{k=1}^{m} -\beta_{0,k}\frac{d}{dz}\log\prod_{\nu=0}^{\infty}\left(1-\mu^{2\nu}z/z_{0,k}\right)\left(1-\mu^{2(\nu+1)}z_{0,k}/z\right)\\
&\quad + \sum_{k=1}^{n}\beta_{1,k}\frac{d}{dz}\log\prod_{\nu=0}^{\infty}\left(1-\mu^{2(\nu+1)}z/z_{1,k}\right)\left(1-\mu^{2\nu}z_{1,k}/z\right).
\end{aligned}$$

Using (2), we have

$$\frac{d}{dz}\log f'(z) = \frac{d}{dz}\sum_{k=1}^{m} -\beta_{0,k}\log\Theta\left(\frac{z}{\mu z_{0,k}}\right) + \frac{d}{dz}\sum_{k=1}^{n}\beta_{1,k}\log\Theta\left(\frac{\mu z}{z_{1,k}}\right).$$

Integrating and exponentiating yields the formula

$$f'(z) = A \prod_{k=1}^{m} \left[\Theta\left(\frac{z}{\mu z_{0,k}} \right) \right]^{-\beta_{0,k}} \prod_{k=1}^{n} \left[\Theta\left(\frac{\mu z}{z_{1,k}} \right) \right]^{\beta_{1,k}}, \qquad z \in \mathcal{A}_{\mu,1},$$

and this completes the derivation of (1).

Remark. In retrospect the appearance of the theta function (2) seems a natural consequence of the familiar connection between an annulus and a (period) rectangle via the logarithm. Indeed, it seems quite likely that the kernel in the Villat integral formula for the annulus provided the initial link to elliptic functions [Kom45].

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