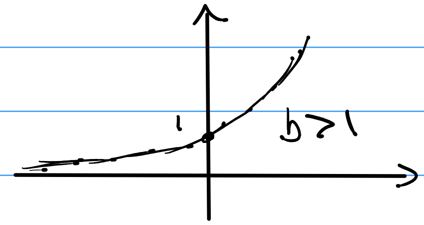
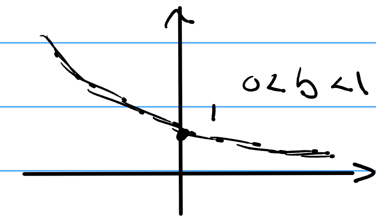
Math 242

Ch6 $f(x) = b^x$





Domain: all reals

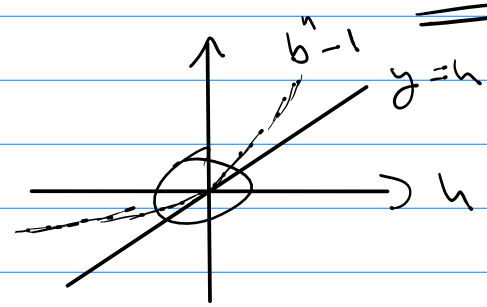
Range: $0 < b^x$, $(0, +\infty)$

Slope of b^x

$$D_x[b^x] = \lim_{h\to 0} \frac{\overbrace{b^{x+h}}^{b^x b^h} - b^x}{h}$$

$$D_x[b^x] = b^x \underbrace{\left[\lim_{h\to 0} \frac{b^h - 1}{h}\right]}_{\text{const.}}$$

$$\lim_{h\to 0} \frac{b^h - 1}{h}$$



if $b = e$

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

So

$$\boxed{f(x) = e^x \qquad D_x[e^x] = e^x}$$

$$\boxed{\int e^x\,dx = e^x + C}$$

ex $D_x\left[\sin(x^2) + e^{x^2}\right]$

$$= \cos(x^2)(2x) + e^{x^2}(2x)$$

$$= \boxed{2x\cos(x^2) + 2xe^{x^2}}$$

ex $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du$

$u = 1+e^{2x}$

$du = 2e^{2x}\,dx$

$$= u^{1/2} + C$$

$$= \boxed{\sqrt{1+e^{2x}} + C}$$

ex $\displaystyle\int \left(e^x e^{e^x}\right)dx = \int e^u\,du = e^u + C$

$u = e^x$

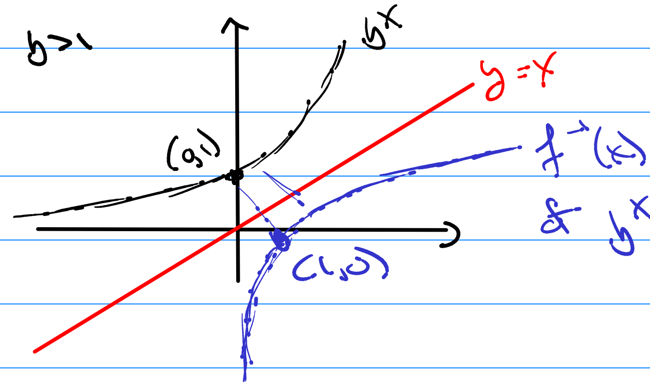
$du = e^x\,dx$

$$= \boxed{e^{e^x} + C}$$

6.3 $f(x) = b^x$ it is a one-to-one function

→ $f^{-1}(x)$ exists

b/c	$f(x)=b^x$	f^{-1}
domain	$(-\infty,\infty)$	$(0,\infty)$
range	$(0,\infty)$	$(-\infty,\infty)$



Find $f^{-1}(x)$?

① $y = b^x$

② $x = b^y \xrightarrow{\text{algebra}} y = f^{-1}(x)$ $\log_b(x)$

Def: the inverse function of $f(x) = b^x$

is: $f^{-1}(x) = \log_b(x)$

where: $\boxed{\log_b(x) = y \text{ iff } x = b^y}$

ex $\log_2 16 = y$ iff $16 = 2^y$

$\boxed{y=4}$ b/c $2^4 = 16$

Solve: $\log_3(2x+1) = 2$

iff $(2x+1) = 3^2$

$2x+1 = 9$

$2x = 8$

$\boxed{x=4}$

Inverse Properties

$f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

So $\log_b(b^x) = x$

$b^{\log_b x} = x$

ex $e^{7-4x} = 6$

$\log_e e^{7-4x} = \log_e 6$

$7-4x = \log_e 6$

$\boxed{x = \dfrac{7-\log_e 6}{4}}$

ex $2^{3x^2-1} = 8 \to 2^{3x^2-1} = 2^3$

$\log_2(2^{3x^2-1}) = \log_2(8)$

$3x^2-1 = 3$

$x = \pm\sqrt{4/3}$

ex $4^{3x^2-1} = 8$

$\log_4(4^{3x^2-1}) = \log_4(8) \to 4^{?} = 8$

$4^{3/2} = 8$

$3x^2-1 = 3/2$

$3x^2 = 5/2$

$x = \pm\sqrt{5/6}$

$b^x b^y = b^{x+y}$ $\qquad \log_b(xy) = \log_b x + \log_b y$

$\frac{b^x}{b^y} = b^{x-y}$ $\qquad \log_b(\frac{x}{y}) = \log_b x - \log_b y$

$(b^x)^r = b^{xr}$ $\qquad \log_b(x^r) = r\log_b x$

Special bases

$b = e$ (natural number)

e^x vs $\log_e x$

$\to \ln(x)$

$b = 10$ (common)

10^x vs $\log_{10} x$

$\to \log x$

$\boxed{\text{Apps}}$

① $2\ln x + 3\ln y - \ln z$

$= \ln x^2 + \ln y^3 - \ln z = \ln\left(\frac{x^2 y^3}{z}\right)$

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② geometric mean $(x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{1/4}$

$\boxed{\text{use}}$ $e^{\ln x} = x$

$(x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{1/4} = e^{\ln(x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{1/4}} = e^{\frac{1}{4}\ln(x_1 \cdot x_2 \cdot x_3 \cdot x_4)}$

$= e^{\frac{\ln(x_1) + \ln(x_2) + \ln(x_3) + \ln(x_4)}{4}}$

$\boxed{\text{Change of base}}$ $b^x \quad \log_b(x) \qquad b \neq e$

$b \neq 10$

$D_x\{b^x\} = b^x \underbrace{\left\{\lim_{h\to 0} \frac{b^h - 1}{h}\right\}}_{D} = \ln(b)$

Change of base b into $b = e$.

① $\log_b x = \frac{1}{\ln b} \cdot \ln(x)$

② $b^x = e^{\ln(b^x)} = e^{x \cdot \ln(b)}$

(5) $2^{x+1} = e^{(x+1)\ln 2}$ $\qquad \log_2(3x-5) = \frac{1}{\ln(2)} \ln(3x-5)$

Derivatives

① $D_x[e^x] = e^x$

② $D_x[b^x] = D_x[e^{x\ln b}] = e^{x\ln b} \cdot \ln b$

$= b^x [\ln b]$

$f^{-1}?$

$$D_x[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

③ $D_x[\ln x] = \dfrac{1}{e^{\ln(x)}} = \dfrac{1}{x}$

$f(x) = e^x \rightarrow f'(x) = e^x$

$f^{-1}(x) = \ln(x)$

Note: on ① $D_x[e^x] = e^x$ Domain: $(-\infty, \infty)$

on ③ $D_x[\ln(x)] = \frac{1}{x}$ Domain: $(0, \infty)$

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④ $D_x[\log_b(x)] = \frac{1}{\ln(b)} \frac{1}{x}$ Domain: $(0, \infty)$

b/c $\frac{1}{\ln(b)} \ln(x)$

Antiderivatives: ① $\int e^x\,dx = e^x + C$

② $\int b^x\,dx = \frac{1}{\ln(b)} b^x + C$

③ $\int \frac{1}{x}\,dx = \ln|x| + C$

$(-\infty, 0)$

Consider: $\ln|x| = \begin{cases} \ln(x) & x>0 \\ \ln(-x) & x<0 \end{cases}$

$$D_x[\ln|x|] = \begin{cases} D_x[\ln(x)] \\ D_x[\ln(-x)] \end{cases} = \begin{cases} \frac{1}{x} \\ \frac{1}{-x}\cdot(-1) = \frac{1}{x} \end{cases}$$

Ex $D_x[x^5 + 5^x] = D_x[x^5] + D_x[5^x]$

$= 5x^4 + \ln(5)\,5^x$

Ex $D_x\left[\ln\left(\sqrt{\frac{a^2-x^2}{a^2+x^2}}\right)\right]$

$$= \frac{1}{\sqrt{\frac{a^2-x^2}{a^2+x^2}}} \cdot D_x\left[\sqrt{\frac{a^2-x^2}{a^2+x^2}}\right]$$

$$= \sqrt{\frac{a^2+x^2}{a^2-x^2}}\ \frac{1}{2}\left[\frac{a^2-x^2}{a^2+x^2}\right]^{-1/2}\left[\frac{(-2x)(a^2+x^2)-(a^2-x^2)(2x)}{(a^2+x^2)^2}\right]$$

(15) $D_x\left[\ln\left(\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}\right)\right]$

$$\ln\sqrt{\frac{a^2-x^2}{a^2+x^2}} = \frac{1}{2}\ln\frac{a^2-x^2}{a^2+x^2} = \frac{1}{2}\left[\ln(a^2-x^2)-\ln(a^2+x^2)\right]$$

$$= D_x\left[\frac{1}{2}\ln(a^2-x^2) - \frac{1}{2}\ln(a^2+x^2)\right]$$

$$= \frac{1}{2}\,\frac{1}{a^2-x^2}\cdot(-2x) - \frac{1}{2}\,\frac{1}{a^2+x^2}\cdot(2x)$$

$$= \boxed{\frac{x}{x^2-a^2} - \frac{x}{x^2+a^2}}$$

Logarithmic Differentiation

(ex) $D_x[x^x]$ Note

x^{const}	x^2
const^x	e^x

Know these

Note

$$y = x^x$$

$$D_x\{y\} = D_x[x^x]$$

$$y' = D_x\{x^x\}$$

Consider $\quad \ln(y) = \ln(x^x)$

[now]

Implicit Derivatives

$$\ln(y) = x\ln(x)$$

$$\frac{1}{y}y' = (1)\ln(x) + x\left(\frac{1}{x}\right)$$

$$y' = y\left[\ln(x) + 1\right]$$

$$y' = x^x\left[\ln(x) + 1\right]$$

$$\boxed{D_x\left[x^x\right] = x^x\left[\ln(x) + 1\right]}$$

Know

$\boxed{f(x) = e^x}$ $\boxed{f(x) = \ln(x)}$

1. Derivatives
2. Antiderivatives