Polyhedra are the surfaces which are analogous to polygons. A polyhedron is a surface made by attaching polygons together along their edges. No edge can be shared by more than two faces. A polyhedron is closed if it is the surface of a solid.

When one speaks of the intrinsic geometry of a polyhedron one means everything that can be determined by making measurements without ever leaving the polyhedron. For instance, one defines the distance between two points $x$ and $y$ in a polyhedron to be the length of the shortest path connecting $x$ and $y$ which stays on the polyhedron.

While the yellow and blue faces of the red polyhedron pictured are at an angle to each other, we can copy the yellow and blue faces in the plane as shown, keeping them attached along their common edge. A little thought should convince the reader that the shortest path between points of these two faces is easier to find using this planar representation, one just uses a straight line segment. This trick also shows one how to continue a straight line segment in one face into an adjacent face. Extending line segments from one face to the next this way does not yield a straight line in three dimensional space. However, it gives a straight line in the intrinsic geometry of our polyhedron.

Curvature. Assume that $v$ is a vertex for faces $F_1, \ldots, F_k$. Assume that for each $j \in 1, \ldots, k$ the angle of the corner of $F_j$ at $v$ is $\alpha_j$. The curvature of the polyhedron at $v$ is defined to be $2\pi - (\alpha_1 + \cdots + \alpha_k)$.

I. Every face of the red stellated icosahedron above is an equilateral triangle. Decide which vertices have positive, negative or zero curvature.

II. If $v$ is a vertex of some polyhedron and $C_r$ is the curve comprised of all points a fixed small distance $r$ from $v$, find the arc length of $C_r$ in terms of $r$ and the curvature of $v$.

Remark. If $v$ is positively curved then $C_r$ is too short to be a circle in the plane, which forces the faces attached to $v$ to be constricted together. If $v$ is negatively curved then $C_r$ is too long to be a planar circle, causing the attached faces to look “floppy”—there is too much stuff attached to $v$ to fit into a plane.

III. Assume two straight lines pass on opposite sides of a vertex $v$ of some polyhedron. Assume that before they passed $v$ they angled away from each other at angle $\theta$ (with the convention that $\theta$ is zero if the lines are parallel, and is negative if the lines converge toward each other). Find the angle at which the lines are heading away from each other after passing $v$. State your answer in terms of $\theta$ and the curvature of $v$. State in plain terms what effect passing on opposite sides of $v$ has on a pair of lines in the following cases: $v$ is positively curved, $v$ has zero curvature, $v$ is negatively curved.

Parallel Transport. If some figure, such as a copy of the letter $\beta$, was placed on the yellow face of the above figure, one could see how to move the figure from the yellow face to the blue face without rotating as follows. Draw copies of the yellow and blue faces in the plane as above. One then slides the figure from the copy of the yellow face to the copy of the blue face in the plane. Finally one copies the result back to the original blue face on the polyhedron. Moving a figure in this way is called parallel transport.
IV. An astonishing thing happens if we move a figure along a polyhedron by parallel transport in a loop around a vertex: when the figure arrives back at its starting place it has rotated.\footnote{This is easy to convince yourself of on a cube.} If a figure moves by parallel transport clockwise around some vertex, what is the angle by which the figure has rotated when it arrives back at its starting point? State your answer in terms of the curvature of the polyhedron at $v$, measuring the angle of rotation of the figure in the counterclockwise direction.

V. To the right is shown approximations of two smooth surfaces by polyhedral surfaces. Every vertex of the first surface has slight positive curvature, and every vertex of the second surface has slight negative curvature. If we move a figure around the given loop by parallel transport, decide for each surface whether the figure will rotate in the clockwise or counterclockwise direction. Find a formula for the rotation a figure moved by parallel transport around such a loop will undergo in terms of curvatures of vertices on the surface.\footnote{If possible, try to provide a convincing argument that your answer is correct.}

VI. Finally, if one draws parallel line segments on a face of these two polyhedra, and continues each along the surface, will the resulting curves converge, diverge or stay parallel?

**Submitting Solutions.** Please submit any complete or partial solutions to John Robertson, 364 Jabara Hall, robertson@math.wichita.edu.

You can download this problem at http://www.math.wichita.edu/~robertson/month/