Isometry Groups
of Pseudoriemannian 2-step Nilpotent Lie Groups

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16 Sep 2007

Abstract: While still a semidirect product, the isometry group can be strictly larger than the obvious Riemannian analogue \( I_{\text{aut}} \). In fact, there are three relevant groups of isometries, \( I_{\text{spl}} \leq I_{\text{aut}} \leq I \), and \( I_{\text{spl}} < I_{\text{aut}} < I \) is possible when the center is degenerate. When the center is nondegenerate, \( I_{\text{spl}} = I \).

To appear in Houston J. Math.


\(^1\)Partially supported by Projects XUGA20702B96 and MEC:MTM2005-08757-C04-01, Spain.
\(^2\)Partially supported by MEC:DGES Program SAB1995-0757, Spain.
1 Introduction

While there had not been much published on the geometry of nilpotent Lie groups with a left-invariant Riemannian metric in 1990 [7], the situation is certainly better now; e.g., [12, 13, 18] and the recent survey [9]. However, there is still almost nothing extant about the more general pseudoriemannian case. In particular, the 2-step nilpotent groups are nonabelian and as close as possible to being Abelian, but display a rich variety of new and interesting geometric phenomena. As in the Riemannian case, one of many places where they arise naturally is as groups of isometries acting on horospheres in certain (pseudoriemannian) symmetric spaces. Another is in the Iwasawa decomposition of semisimple groups with the Killing metric, which need not be definite. Here we study the isometry groups of these group spaces.

One motivation for our study was our observation in [4] that there are two nonisometric pseudoriemannian metrics on the Heisenberg group $H_3$, one of which is flat. This is a strong contrast to the Riemannian case in which there is only one (up to positive homothety) and it is not flat. This is not an anomaly, as we shall see later. We were also inspired by the paper of Eberlein [7, 8]. Since the published version is not identical to the preprint, we have cited both where appropriate.

While the geometric properties of Lie groups with left-invariant definite metric tensors have been studied extensively, the same has not occurred for indefinite metric tensors. For example, while the paper of Milnor [14] has already become a classic reference, in particular for the classification of positive definite (Riemannian) metrics on 3-dimensional Lie groups, a classification of the left-invariant Lorentzian metric tensors on these groups became available only relatively recently [4]. Similarly, only a few partial results in the line of Milnor’s study of definite metrics were previously known for indefinite metrics [1, 15]. Moreover, in dimension 3 there are only two types of metric tensors: Riemannian (definite) and Lorentzian (indefinite). But in higher dimensions there are many distinct types of indefinite metrics while there is still essentially only one type of definite metric. This is another reason our work here and in [5] has special interest.

By an inner product on a vector space $V$ we shall mean a nondegenerate, symmetric bilinear form on $V$, generally denoted by $\langle , \rangle$. In particular, we do not assume that it is positive definite. Our convention is that $v \in V$ is timelike if $\langle v, v \rangle > 0$, null if $\langle v, v \rangle = 0$, and spacelike if $\langle v, v \rangle < 0$.

Throughout, $N$ will denote a connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ having center $\mathfrak{z}$ with complement $\mathfrak{v}$. (Recall that 2-step means
We shall use $\langle \cdot , \cdot \rangle$ to denote either an inner product on $n$ or the induced left-invariant pseudoriemannian (indefinite) metric tensor on $N$.

In Section 2, we give the fundamental definitions and examples used in the rest of this paper. The main problem encountered is that the center $z$ of $n$ may be degenerate: it might contain a (totally) null subspace. We shall see that this possible degeneracy of the center causes the essential differences between the Riemannian and pseudoriemannian cases.

Section 3 contains basic information on the isometry group of $N$. In particular, it can be strictly larger in a significant way than in the Riemannian (or pseudoriemannian with nondegenerate center) case. Letting $A$ denote the automorphism group and $I$ the isometry group, set $O = A \cap I$. As the choice might suggest, $O$ is analogous to a pseudorthogonal group [20]. Let $\tilde{O}$ denote the subgroup of $I$ which fixes $1 \in N$, so $I \cong O \rtimes N$ where $N$ acts by left translations. Also set $I^{\text{aut}} = O \rtimes N$. Finally, let $I^{\text{spl}}$ denote the subgroup of $I$ which preserves the splitting $TN = zN \oplus vN$. We show that $I^{\text{spl}} \leq I^{\text{aut}} \leq I$ and spend much of the section constructing examples to show that $I^{\text{spl}} < I^{\text{aut}} < I$ is possible when the center is degenerate. When the center is nondegenerate, $I^{\text{spl}} = I$.

We recall [5] some basic facts about 2-step nilpotent Lie groups. As with all nilpotent Lie groups, the exponential map $\exp : n \to N$ is surjective. Indeed, it is a diffeomorphism for simply connected $N$; in this case we shall denote the inverse by $\log$. The Baker-Campbell-Hausdorff formula takes on a particularly simple form in these groups:

$$\exp(x) \exp(y) = \exp(x + y + \frac{1}{2}[x,y]). \quad (1.1)$$

Letting $L_n$ denote left translation by $n \in N$, we have the following description.

**Lemma 1.1** Let $n$ denote a 2-step nilpotent Lie algebra and $N$ the corresponding simply connected Lie group. If $x, a \in n$, then

$$\exp_{x*}(a_x) = L_{\exp(x)*} \left( a + \frac{1}{2}[a, x] \right)$$

where $a_x$ denotes the initial velocity vector of the curve $t \mapsto x + ta$. \hfill $\square$

**Corollary 1.2** In a pseudoriemannian 2-step nilpotent Lie group, the exponential map preserves causal character. Alternatively, 1-parameter subgroups are curves of constant causal character.

**Proof:** For the 1-parameter subgroup $c(t) = \exp(ta), \dot{c}(t) = \exp_{ta*}(a) = L_{\exp(ta)*}a$ and left translations are isometries. \hfill $\square$
Of course, 1-parameter subgroups need not be geodesics, as simple examples show [5].

For 2-step nilpotent Lie groups, things work nicely as shown by this result first published by Guediri [10].

**Theorem 1.3** On a 2-step nilpotent Lie group, all left-invariant pseudoriemannian metrics are geodesically complete.

He also provided an explicit example of an incomplete metric on a 3-step nilpotent Lie group.

We refer to [5] for complete calculations of explicit formulas for the connection, curvatures, covariant derivative, etc.

## 2 Definitions and Examples

In the Riemannian (positive-definite) case, one splits $n = z \oplus v = z \oplus z^\perp$ where the superscript denotes the orthogonal complement with respect to the inner product $\langle , \rangle$. In the general pseudoriemannian case, however, $z \oplus z^\perp \neq n$. The problem is that $z$ might be a degenerate subspace; i.e., it might contain a null subspace $U$ for which $U \subseteq U^\perp$.

Thus we shall have to adopt a more complicated decomposition of $n$. Observe that if $z$ is degenerate, the null subspace $U$ is well defined invariantly. We shall use a decomposition

$$n = z \oplus v = U \oplus Z \oplus V \oplus E$$

in which $z = U \oplus Z$ and $v = V \oplus E$, $U$ and $V$ are complementary null subspaces, and $U^\perp \cap V^\perp = Z \oplus E$. Although the choice of $W$ is not well defined invariantly, once a $W$ has been chosen then $Z$ and $E$ are well defined invariantly. Indeed, $Z$ is the portion of the center $z$ in $U^\perp \cap V^\perp$ and $E$ is its orthocomplement in $U^\perp \cap V^\perp$. This is a Witt decomposition of $n$ given $U$ as described in [17, p. 37f], easily seen by noting that $(U \oplus V)^\perp = Z \oplus E$, adapted to the special role of the center in $n$.

We now fix a choice of $W$ (and therefore $Z$ and $E$) to be maintained throughout this paper. Whenever the effect of this choice is to be considered, it will be done so explicitly.

Having fixed $W$, observe that the inner product $\langle , \rangle$ provides a dual pairing between $U$ and $W$; i.e., isomorphisms $U^\ast \cong W$ and $U \cong W^\ast$. Thus the choice of a basis $\{u_i\}$ in $U$ determines an isomorphism $U \cong W$ (via the dual basis $\{v_i\}$ in $W$).
In addition to the choice of \( \mathfrak{U} \), we now also fix a basis of \( \mathfrak{U} \) to be maintained throughout this paper. Whenever the effect of this choice is to be considered, it will also be done so explicitly.

We shall also need to use an involution \( \iota \) that interchanges \( \mathfrak{U} \) and \( \mathfrak{V} \) by this isomorphism and which reduces to the identity on \( \mathfrak{Z} \oplus \mathfrak{E} \) in the Riemannian (positive-definite) case. The choice of such an involution is not significant \([5]\). In terms of chosen orthonormal bases \( \{ z_\alpha \} \) of \( \mathfrak{Z} \) and \( \{ e_a \} \) of \( \mathfrak{E} \),

\[
\iota(u_i) = v_i, \quad \iota(v_i) = u_i, \quad \iota(z_\alpha) = \varepsilon_\alpha z_\alpha, \quad \iota(e_a) = \bar{\varepsilon}_a e_a,
\]

where, as usual,

\[
\langle u_i, v_i \rangle = 1, \quad \langle z_\alpha, z_\alpha \rangle = \varepsilon_\alpha, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.
\]

Then \( \iota(\mathfrak{U}) = \mathfrak{V}, \ i(\mathfrak{V}) = \mathfrak{U}, \ i(\mathfrak{Z}) = \mathfrak{Z}, \ i(\mathfrak{E}) = \mathfrak{E} \) and \( \iota^2 = I \). With respect to this basis \( \{ u_i, z_\alpha, v_i, e_a \} \) of \( \mathfrak{n} \), \( \iota \) is given by the following matrix:

\[
\begin{bmatrix}
1 & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & 1 \\
\varepsilon_1 & \cdots & 0 \\
0 & \ddots & \ddots \\
0 & \cdots & \varepsilon_r \\
1 & \cdots & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & 1 \\
0 & 0 & \cdots & \varepsilon_1 \\
0 & 0 & \cdots & \ddots \\
0 & \cdots & \varepsilon_s
\end{bmatrix}
\]

We note that this is also the matrix of \( \langle \cdot, \cdot \rangle \) on the same basis; however, \( \iota \) is a linear transformation, so it will transform differently with respect to a change of basis.

It is obvious that \( \iota \) is self adjoint with respect to the inner product,

\[
\langle \iota x, y \rangle = \langle x, \iota y \rangle, \quad x, y \in \mathfrak{n},
\]

so \( \iota \) is an isometry of \( \mathfrak{n} \). (However, it does not integrate to an isometry of \( \mathfrak{N} \); see Example 2.13.) Moreover,

\[
\langle x, \iota x \rangle = 0 \text{ if and only if } x = 0, \quad x \in \mathfrak{n}.
\]
Thus for all \( y \in \mathfrak{n} \), \( \text{ad}_a^\dagger y \) maps \( \mathfrak{V} \oplus \mathfrak{E} \) to \( \mathfrak{U} \oplus \mathfrak{E} \). Moreover, for all \( u \in \mathfrak{U} \) we have \( \text{ad}_u^\dagger = 0 \) and for all \( e \in \mathfrak{E} \) also \( \text{ad}_e^\dagger e = 0 \). Following [11, 7, 8], we next define the operator \( j \). Note the use of the involution \( \iota \) to obtain a good analogy to the Riemannian case.

**Definition 2.1** The linear mapping

\[
j : \mathfrak{U} \oplus \mathfrak{Z} \rightarrow \text{End} (\mathfrak{V} \oplus \mathfrak{E})
\]

is given by

\[
j(a)x = \iota \text{ad}_a^\dagger \iota a.
\]

Equivalently, one has the following characterization:

\[
\langle j(a)x, \iota y \rangle = \langle [x,y], \iota a \rangle, \quad a \in \mathfrak{U} \oplus \mathfrak{Z}, \ x,y \in \mathfrak{V} \oplus \mathfrak{E}.
\]

It turns out [5] that this map (together with the Lie algebra structure of \( \mathfrak{n} \)) determines the geometry of \( N \) just as in the Riemannian case [8]. The operator \( j \) is \( \iota \)-skewadjoint with respect to the inner product \( \langle , \rangle \).

**Proposition 2.2** For every \( a \in \mathfrak{U} \oplus \mathfrak{Z} \) and all \( x,y \in \mathfrak{V} \oplus \mathfrak{E} \),

\[
\langle j(a)x, \iota y \rangle + \langle \iota x, j(a)y \rangle = 0.
\]

The Riemannian version [11, 8] is easily obtained as the particular case when we assume \( \mathfrak{U} = \mathfrak{V} = \{0\} \) and the inner product on \( \mathfrak{Z} \oplus \mathfrak{E} \) is positive definite (i.e., \( \varepsilon_\alpha = \bar{\varepsilon}_\alpha = 1 \)); then \( \iota = I \) and we recover the definitions of [11]. This recovery of the Riemannian case continues in all that follows.

Let \( x, y \in \mathfrak{n} \). Recall [2, 5] that homaloidal planes are those for which the numerator \( \langle R(x,y)y, x \rangle \) of the sectional curvature \( K(x,y) \) vanishes. This notion is useful for degenerate planes tangent to spaces that are not of constant curvature.

**Theorem 2.3** All central planes are homaloidal: \( R(z,z')z'' = R(u,x)y = R(x,y)u = 0 \) for all \( z,z',z'' \in \mathfrak{Z}, \ u \in \mathfrak{U}, \) and \( x,y \in \mathfrak{n} \). Thus the nondegenerate part of the center is flat:

\[
K(z,z') = 0.
\]

This recovers [8, (2.4), item c)] in the Riemannian case.

In view of this result, we extend the notion of flatness to possibly degenerate submanifolds.
Definition 2.4 A submanifold of a pseudoeuclidean manifold is flat if and only if every plane tangent to the submanifold is homaloidal.

Corollary 2.5 The center $Z$ of $N$ is flat.

Corollary 2.6 The only $N$ of constant curvature are flat.

The degenerate part of the center can have a profound effect on the geometry of the whole group.

Theorem 2.7 If $[n,n] \subseteq \mathfrak{U}$ and $\mathfrak{E} = \{0\}$, then $N$ is flat.

Among these spaces, those that also have $\mathfrak{F} = \{0\}$ (which condition itself implies $[n,n] \subseteq \mathfrak{U}$) are fundamental, with the more general ones obtained by making nondegenerate central extensions. It is also easy to see that the product of any flat group with a nondegenerate abelian factor is still flat.

This is the best possible result in general. Using weaker hypotheses in place of $\mathfrak{E} = \{0\}$, such as $[\mathfrak{V},\mathfrak{V}] = \{0\} = [\mathfrak{E},\mathfrak{E}]$, it is easy to construct examples which are not flat.

Corollary 2.8 If $\dim Z \geq \lceil \frac{n}{2} \rceil$, then there exists a flat metric on $N$.

Here $\lceil r \rceil$ denotes the least integer greater than or equal to $r$.

Before continuing, we pause to collect some facts about the condition $[n,n] \subseteq \mathfrak{U}$ and its consequences.

Remarks 2.9 Since it implies $j(z) = 0$ for all $z \in \mathfrak{F}$, this latter is possible with no pseudoeuclidean de Rham factor, unlike the Riemannian case.

Also, it implies $j(u)$ interchanges $\mathfrak{V}$ and $\mathfrak{E}$ for all $u \in \mathfrak{U}$ if and only if $[\mathfrak{V},\mathfrak{V}] = [\mathfrak{E},\mathfrak{E}] = \{0\}$. Examples are the Heisenberg group and the groups $H(p,1)$ for $p \geq 2$ with null centers.

Finally we note it implies that, for every $u \in \mathfrak{U}$, $j(u)$ maps $\mathfrak{V}$ to $\mathfrak{V}$ if and only if $j(u)$ maps $\mathfrak{E}$ to $\mathfrak{E}$ if and only if $[\mathfrak{V},\mathfrak{E}] = \{0\}$.

Recall that the Lie algebra $\mathfrak{n}$ is said to be nonsingular if and only if $\text{ad}_x$ maps $\mathfrak{n}$ onto $\mathfrak{F}$ for every $x \in \mathfrak{n} - \mathfrak{F}$. As in [8], we immediately obtain

Lemma 2.10 The 2-step nilpotent Lie algebra $\mathfrak{n}$ is nonsingular if and only if for every $a \in \mathfrak{U} \oplus \mathfrak{F}$ the maps $j(a)$ are nonsingular, for every inner product on $\mathfrak{n}$, every choice of $\mathfrak{V}$, every basis of $\mathfrak{U}$, and every choice of $i$.

Next we consider some examples of these Lie groups.
Example 2.11  The usual inner products on the 3-dimensional Heisenberg algebra $h_3$ may be described as follows. On an orthonormal basis $\{z,e_1,e_2\}$ the structure equation is

$$[e_1,e_2] = z$$

with nontrivial inner products

$$\varepsilon = \langle z,z \rangle, \quad \bar{\varepsilon}_1 = \langle e_1,e_1 \rangle, \quad \bar{\varepsilon}_2 = \langle e_2,e_2 \rangle.$$ 

We find the nontrivial adjoint maps as

$$\text{ad}^\dagger_{e_1} z = \varepsilon \bar{\varepsilon}_2 e_2, \quad \text{ad}^\dagger_{e_2} z = -\varepsilon \bar{\varepsilon}_1 e_1$$

with the rest vanishing. On the basis $\{e_1,e_2\}$,

$$j(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so $j(z)^2 = -I_2$. Moreover, a direct computation shows that

$$j(a)^2 = -\langle a,\iota a \rangle I_2, \quad \text{for all } a \in \mathfrak{z}.$$ 

This construction extends to the generalized Heisenberg groups $H(p,1)$ of dimension $2p+1$ with non-null center of dimension 1 generated by $z$, and we find

$$j(z) = \begin{bmatrix} 0 & -I_p \\ I_p & 0 \end{bmatrix}.$$

Example 2.12  In [4] we presented an inner product on the 3-dimensional Heisenberg algebra $h_3$ for which the center is degenerate. First we recall that on an orthonormal basis $\{e_1,e_2,e_3\}$ with signature $(+−−)$ the structure equations are

$$[e_3,e_1] = \frac{1}{2}(e_1 - e_2),$$

$$[e_3,e_2] = \frac{1}{2}(e_1 - e_2),$$

$$[e_1,e_2] = 0.$$ 

The center $\mathfrak{z}$ is the span of $e_1 - e_2$ and is in fact null.

We take a new basis $\{u = \frac{1}{\sqrt{2}}(e_2 - e_1), v = \frac{1}{\sqrt{2}}(e_2 + e_1), e = e_3\}$ and find the structure equation

$$[v,e] = u.$$ 

Generalizing slightly, we take the nontrivial inner products to be

$$\langle u,v \rangle = 1 \quad \text{and} \quad \langle e,e \rangle = \bar{\varepsilon}.$$ 

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We find the nontrivial adjoint maps as
\[
\text{ad}_v^\dagger v = \bar{\varepsilon}e, \quad \text{ad}_e^\dagger v = -u.
\]
On the basis \{v, e\},
\[
j(u) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]
so \(j(u)^2 = -I_2\). Again, a direct computation shows that \(j(a)^2 = -\langle a, \iota a \rangle I_2\) for all \(a \in \mathfrak{H}\).

Again, this construction extends to \(H(p, 1)\) with null center generated by \(u\) and we find
\[
j(u) = \begin{bmatrix} 0 & -I_p \\ I_p & 0 \end{bmatrix}.
\]
According to [4], these are all the Lorentzian inner products on \(\mathfrak{h}_3\) up to homothety.

**Example 2.13** For the simplest quaternionic Heisenberg algebra of dimension 7, we may take a basis \{\(u_1, u_2, z, v_1, v_2, e_1, e_2\)\} with structure equations
\[
[e_1, e_2] = z \quad [v_1, v_2] = z
\]
\[
[e_1, v_1] = u_1 \quad [e_2, v_1] = u_2
\]
\[
[e_1, v_2] = u_2 \quad [e_2, v_2] = -u_1
\]
and nontrivial inner products
\[
\langle u_i, v_j \rangle = \delta_{ij}, \quad \langle z, z \rangle = \varepsilon, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.
\]
As usual, each \(\varepsilon\)-symbol is \(\pm 1\) independently (this is a combined null and orthonormal basis), so the signature is \((+ - - \varepsilon \bar{\varepsilon}_1 \bar{\varepsilon}_2)\).

The nontrivial adjoint maps are
\[
\text{ad}_{e_1}^\dagger z = \varepsilon u_2 \quad \text{ad}_{e_1}^\dagger z = \varepsilon \bar{\varepsilon}_2 e_2
\]
\[
\text{ad}_{e_1}^\dagger v_1 = -\bar{\varepsilon}_1 e_1 \quad \text{ad}_{e_1}^\dagger v_1 = u_1
\]
\[
\text{ad}_{e_1}^\dagger v_2 = -\bar{\varepsilon}_2 e_2 \quad \text{ad}_{e_1}^\dagger v_2 = u_2
\]
\[
\text{ad}_{e_2}^\dagger z = -\varepsilon u_1 \quad \text{ad}_{e_2}^\dagger z = -\varepsilon \bar{\varepsilon}_1 e_1
\]
\[
\text{ad}_{e_2}^\dagger v_1 = \bar{\varepsilon}_2 e_2 \quad \text{ad}_{e_2}^\dagger v_1 = -u_2
\]
\[
\text{ad}_{e_2}^\dagger v_2 = -\bar{\varepsilon}_1 e_1 \quad \text{ad}_{e_2}^\dagger v_2 = u_1
\]
For \(j\) on the basis \{\(v_1, v_2, e_1, e_2\)\} we obtain
\[
j(u_1) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\]
so \( j(u_1)^2 = -I_4 \),

\[
j(u_2) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}
\]

so \( j(u_2)^2 = -I_4 \), and

\[
j(z) = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

so \( j(z)^2 = -I_4 \). Again, a direct computation shows that \( j(a)^2 = -\langle a, \iota a \rangle I_4 \) for all \( a \in \mathfrak{U} \oplus 3 \).

Once again, this construction may be extended to a nondegenerate center, to a degenerate center with a null subspace of dimensions 1 or 3, and to quaternionic algebras of all dimensions.

Sectional curvatures for this group are

\[
\langle R(v_1, v_2)v_2, v_1 \rangle = -(\bar{\varepsilon}_1 + \frac{3}{4}\bar{\varepsilon}), \\
\langle R(v, e)v, e \rangle = \langle R(z, v)v, z \rangle = 0, \\
K(z, e_1) = K(z, e_2) = \frac{1}{4}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2, \\
K(e_1, e_2) = -\frac{3}{4}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2.
\]

Thus \( \iota \) cannot integrate to an isometry of \( N \) in general, as mentioned after equation (2.1). Isometries must preserve vanishing of sectional curvature, and an integral of \( \iota \) would interchange homaloidal and nonhomaloidal planes in this example.

**Example 2.14** For the generalized Heisenberg group \( H(1, 2) \) of dimension 5 we take the basis \( \{ u, z, v, e_1, e_2 \} \) with structure equations

\[
[e_1, e_2] = z \\
[v, e_2] = u
\]

and nontrivial inner products

\[
\langle u, v \rangle = 1, \quad \langle z, z \rangle = \varepsilon, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.
\]

The signature is \((+ - \varepsilon \bar{\varepsilon}_1 \bar{\varepsilon}_2)\). We find the nontrivial adjoint maps.

\[
\text{ad}^\dagger_v v = \bar{\varepsilon}_2 e_2 \quad \text{ad}^\dagger_{e_1} z = \varepsilon \bar{\varepsilon}_2 e_2 \\
\text{ad}^\dagger_{e_2} v = -u \quad \text{ad}^\dagger_{e_2} z = -\varepsilon \bar{\varepsilon}_1 e_1
\]
On the basis \{v, e_1, e_2\},
\[
j(u) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]
so \(j(u)^2 \cong 0 \oplus -I_2\), and
\[
j(z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}
\]
with \(j(z)^2 = 0 \oplus -I_2\).

This construction, too, extends to the generalized Heisenberg groups \(H(1,p)\) with \(p \geq 2\). The dimension is again \(2p + 1\), but now the center has dimension \(p\). In each case, the \(j\)-endomorphisms have rank 2 with a similar appearance.

3 Isometry group

We begin with a general result about nilpotent Lie groups.

Lemma 3.1 Let \(N\) be a connected, nilpotent Lie group with a left-invariant metric tensor. Any isometry of \(N\) that is also an inner automorphism must be the identity map.

Proof: Consider an isometry of \(N\) which is also the inner automorphism determined by \(g \in N\). Then Ad\(^g\) is a linear isometry of \(n\). The Lie group exponential map is surjective, so there exists \(x \in n\) with \(g = \exp(x)\). Then
\[
\text{Ad}\,g = e^{\text{ad}\,x}.
\]
Now ad\(^x\) is nilpotent so all its eigenvalues are 0. Thus all the eigenvalues of Ad\(^g\) are 1, so it is unipotent. It now follows that Ad\(^g\) is the identity. Thus \(g\) lies in the center of \(N\) and the isometry of \(N\) is the identity. \(\square\)

Now we give some general information about the isometries of \(N\). Letting Aut\((N)\) denote the automorphism group of \(N\) and I\((N)\) the isometry group of \(N\), set O\((N) = \text{Aut}(N) \cap I(N)\). In the Riemannian case, I\((N) = O(N) \ltimes N\), the semidirect product where \(N\) acts as left translations. We have chosen the notation O\((N)\) to suggest an analogy with the pseudoeuclidean case in which this subgroup is precisely the (general, including reflections) pseudorthogonal group. According to Wilson [20], this analogy is good for any nilmanifold (not necessarily 2-step).
To see what is true about the isometry group in general, first consider the (left-invariant) splitting of the tangent bundle $TN = \mathfrak{z}N \oplus \mathfrak{v}N$.

**Definition 3.2** Denote by $I^{spl}(N)$ the subgroup of the isometry group $I(N)$ which preserves the splitting $TN = \mathfrak{z}N \oplus \mathfrak{v}N$. Further, let $I^{aut}(N) = O(N) \ltimes N$, where $N$ acts by left translations.

**Proposition 3.3** If $N$ is a simply-connected, 2-step nilpotent Lie group with left-invariant metric tensor, then $I^{spl}(N) \leq I^{aut}(N)$.

**Proof:** Similarly to the observation by Eberlein [8], it is now easy to check that the relevant part of Kaplan’s proof for Riemannian $N$ of $H$-type [11] is readily adapted and extended to our setting, with the proviso that it is easier just to replace his expressions $j(a)x$ with $\text{ad}^\dagger a$ instead of trying to convert to our $j$. □

We shall give an example to show that $I^{spl} < I^{aut}$ is possible when $\mathfrak{u} \neq \{0\}$.

When the center is degenerate, the relevant group analogous to a pseudo-orthogonal group may be larger.

**Proposition 3.4** Let $\widetilde{O}(N)$ denote the subgroup of $I(N)$ which fixes $1 \in N$. Then $I(N) \cong \widetilde{O}(N) \ltimes N$, where $N$ acts by left translations. □

The proof is obvious from the definition of $\widetilde{O}$. It is also obvious that $O \leq \widetilde{O}$. We shall give an example to show that $O < \widetilde{O}$, hence $I^{aut} < I$, is possible when the center is degenerate.

Thus we have three groups of isometries, not necessarily equal in general: $I^{spl} \leq I^{aut} \leq I$. When the center is nondegenerate ($\mathfrak{u} = \{0\}$), the Ricci transformation is block-diagonalizable and the rest of Kaplan’s proof using it now also works.

**Corollary 3.5** If the center is nondegenerate, then $I(N) = I^{spl}(N)$ whence $\widetilde{O}(N) \cong O(N)$. □

We now begin to prepare for the promised examples. Recall the result in [19, 3.57(b)] that for any simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$,

$$\text{Aut}(G) \cong \text{Aut}(\mathfrak{g}) : \alpha \mapsto \alpha_+ |_{\mathfrak{g}}.$$ 

Now, if $\alpha$ is also an isometry, then so is $\alpha_+$. Conversely, any isometric automorphism of $\mathfrak{g}$ comes from an isometric automorphism of $G$ by means of left-invariance (or homogeneity). Therefore

$$O(N) = \text{Aut}(N) \cap I(N) \cong \text{Aut}(\mathfrak{n}) \cap O^p_q$$

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for one of our simply connected groups \( N \) with signature \((p,q)\). So up to isomorphism, we may regard

\[
O(N) \leq O_q^p.
\]

**Example 3.6** Any flat, simply connected \( N \) is a space form. (By Corollary 2.6 no other constant curvature is possible.) Thus the isometry group is known (up to isomorphism) and \( \tilde{O}(N) \cong O_q^p \) for any such \( N \) of signature \((p,q)\).

For the rest of the examples, we need some additional preparations. Let \( h_3 \) denote the 3-dimensional Heisenberg algebra and consider the unique 4-dimensional, 2-step nilpotent Lie algebra \( n_4 = h_3 \times \mathbb{R} \), with basis \( \{v, e, z, u\} \), structure equation \([v, e] = z\), and center \( z = [z, u] \).

A general automorphism of \( n_4 \) as a vector space, \( f: n_4 \to n_4 \), will be given with respect to the basis \( \{v, e, z, u\} \) by a matrix

\[
\begin{bmatrix}
a_1 & b_1 & c_1 & d_1 \\
 a_2 & b_2 & c_2 & d_2 \\
 a_3 & b_3 & c_3 & d_3 \\
 a_4 & b_4 & c_4 & d_4
\end{bmatrix}.
\]

In order for \( f \) to be a Lie algebra automorphism of \( n_4 \), it must preserve the center, and \([f(v), f(e)] = f(z)\) while the other products of images by \( f \) are zero; all this together implies that the matrix of \( f \) must have the form

\[
\begin{bmatrix}
a_1 & b_1 & 0 & 0 \\
 a_2 & b_2 & 0 & 0 \\
 a_3 & b_3 & c_3 & d_3 \\
 a_4 & b_4 & 0 & d_4
\end{bmatrix}
\]

with \( c_3 = a_1 b_2 - a_2 b_1 \neq 0, d_4 \neq 0 \) and arbitrary \( a_3, a_4, b_3, b_4, d_3 \).

Now the matrix of \( f \) can be decomposed as a product of matrices \( M_1 \cdot M_2 \cdot M_3 \cdot M_4 \), where

\[
M_1 = \begin{bmatrix}
a_1 & b_1 & 0 & 0 \\
 a_2 & b_2 & 0 & 0 \\
 A & B & c_3 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix}, \quad
M_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & d_4
\end{bmatrix},
\]

\[
M_3 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & d_3/c_3 \\
 0 & 0 & 0 & 1
\end{bmatrix}, \quad
M_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 C & D & 0 & 1
\end{bmatrix},
\]

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with
\[ A = \frac{a_3 d_4 - a_4 d_3}{d_4}, \quad B = \frac{b_3 d_4 - b_4 d_3}{d_4}, \quad C = \frac{a_4}{d_4}, \quad D = \frac{b_4}{d_4}, \]
which are well defined because \( c_3, d_4 \neq 0 \). In this decomposition of \( f \) we have:
\[ M_1 \in \text{Aut}(\mathfrak{h}_3), \text{ and they fill out } \text{Aut}(\mathfrak{h}_3); \]
\[ M_2 \in \text{Aut}(\mathbb{R}); \]
\[ M_3 \in M \text{ mixes part of the center but fixes } v, e, \text{ and } z; \]
\[ M_4 \text{ fixes } z \text{ and } u \text{ but the image of } [v, e] \text{ is contained in } [v, e, u]. \]
The matrices \( M_4 \) form a group isomorphic to an abelian \( \mathbb{R}^2 \). Note that for such an \( M_4, v \mapsto v + Cu \) and \( e \mapsto e + Du. \)

**Proposition 3.7** The decomposition of \( \text{Aut}(\mathfrak{n}_4) \) is
\[ \text{Aut}(\mathfrak{n}_4) = \text{Aut}(\mathfrak{h}_3) \cdot \text{Aut}(\mathbb{R}) \cdot M \cdot \mathbb{R}^2. \]

Now we compute \( O(N) \) for several simply-connected groups of interest. Specifically, we shall compute \( O(N) \cong O(\mathfrak{n}) \) for the flat and nondegenerate 3-dimensional Heisenberg groups \( H_3 \) and the 4-dimensional groups \( N_4 \cong H_3 \times \mathbb{R} \) with all possible signatures.

**Example 3.8** We begin with the form for automorphisms of \( \mathfrak{h}_3 \) on the basis \( \{v, e, u\} \),
\[ f = \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{bmatrix} \]
with \( c_3 = a_1 b_2 - a_2 b_1 \neq 0 \) and arbitrary \( a_3, b_3 \). For the flat metric tensor we use
\[ \eta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \bar{v} & 0 \\ 1 & 0 & 0 \end{bmatrix} \]
and we need to solve
\[ f^T \eta f = \eta. \]
Using *Mathematica* to help, it is easy to see that we must have \( b_1 = 0 \), whence \( a_1 \neq 0 \) and thus the **Solve** command in fact does give all the solutions. We also obtain \( a_1 = \pm 1 \), so
\[ f = \begin{bmatrix} \pm 1 & 0 & 0 \\ a_2 & 1 & 0 \\ \mp \bar{v} a_2^2/2 & \mp \bar{v} a_2 & \pm 1 \end{bmatrix} \quad (3.1) \]
with arbitrary $a_2$. We note that the determinant is 1, so that this 1-parameter group lies in $SO^1_2$ or $SO^2_1$ according as $\bar{\varepsilon} = \pm 1$, respectively. With some additional work, this is seen to be a group of horocyclic translations (HT in [3]) subjected to a change of basis (rotation and reordering). (Comparing parameters, we have here $a_2 = -t\sqrt{2}$ there.) Note $\dim \tilde{O}(H_3) = 1$ while from Example 3.6 we have $\dim \tilde{O}(H_3) = 3$. We can identify the rest of $\tilde{O}$ from an Iwasawa decomposition: $O$ is the nilpotent part, so the rest of $\tilde{O}$ consists of rotations and boosts.

Many groups with degenerate center have such a subgroup isometrically embedded.

**Proposition 3.9** Assume $\mathfrak{U} \neq \{0\} \neq \mathfrak{E}$ and let $u \in \mathfrak{U}$, $v \in \mathfrak{V}$, such that $\langle u, v \rangle = 1$ and $0 \neq j(u)v = e \in \mathfrak{E}$. Then $j(u)^2 = -I$ on $[v, e]$ if and only if $\mathfrak{h} = [u, v, e]$ is isometric to the 3-dimensional Heisenberg algebra with null center. Consequently, $H = \exp \mathfrak{h}$ is an isometrically embedded, flat subgroup of $N$.

**Proof:** It is easy to see that $j(u)e = -v$; cf. Remarks 2.9. Then, on the one hand, $\langle [v, j(u)v], v \rangle = \langle [v, e], v \rangle$. On the other hand, $\langle [v, j(u)v], v \rangle = -\langle [j(u)v, v], v \rangle = -\langle \text{ad}^1_{j(u)}u, v \rangle = -\langle j(u)^2v, v \rangle = \langle vv, v \rangle = \langle u, v \rangle = 1$. It follows that $[v, e] = u$. The converse follows from Example 2.12. □

**Corollary 3.10** For any $N$ containing such a subgroup,

$$I^{qd}(N) < I^{ad}(N) < I(N).$$

Unfortunately, this class does not include our flat groups of Theorem 2.7 in which $[n, n] \subseteq \mathfrak{U}$ and $\mathfrak{E} = \{0\}$. However, it does include many groups that do not satisfy $[n, n] \subseteq \mathfrak{U}$, such as the simplest quaternionic Heisenberg group of Example 2.13 et seq.

**Remark 3.11** A direct computation shows that on this flat $H_3$ with null center, the only Killing fields with geodesic integral curves are the nonzero scalar multiples of $u$.

Similarly, we can do the 3-dimensional Heisenberg group with nondegenerate center (including certain definite cases). With respect to the orthonormal basis $\{e_1, e_2, z\}$ with structure equation $[e_1, e_2] = z$, we obtain

$$\tilde{O}(H_3) = O(H_3) \cong \begin{cases} O_2 & \text{if } \bar{\varepsilon}_1\bar{\varepsilon}_2 = 1, \\ O_1^1 & \text{if } \bar{\varepsilon}_1\bar{\varepsilon}_2 = -1 \end{cases}$$
of dimension two.

In general, we always can arrange to have \( e_1 \) and \( e_2 \) orthonormal, but we may not be able to keep the structure equation and have a unit vector simultaneously. In the indefinite case, we always can have the z-axis orthogonal to the \( e_1e_2 \)-plane. But in the definite case, the \( e_1e_2 \)-plane need not be orthogonal to the z-axis. It is not yet clear how best to compute \( O(H_3) \) in these cases.

**Example 3.12** Now we consider the flat \( N_4 \cong H_3 \times \mathbb{R} \) with null center. Here we have the basis \( \{v_1, v_2, u_1, u_2\} \) of \( n_4 \) with structure equation \([v_1, v_2] = u_1\), the form for automorphisms

\[
f = \begin{bmatrix} a_1 b_1 & 0 & 0 \\ a_2 b_2 & 0 & 0 \\ a_3 b_3 c_3 d_3 \\ a_4 b_4 & 0 & d_4 \end{bmatrix}
\]

with \( c_3 = a_1 b_2 - a_2 b_1 \neq 0 \), \( d_4 \neq 0 \), and arbitrary \( a_3, a_4, b_3, b_4, d_3 \), and the flat metric tensor

\[
\eta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\]

Again we find \( b_1 = 0 \) and \( a_1 \neq 0 \) and Solve gives all solutions of \( f^T \eta f = \eta \). We obtain

\[
f = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_2 & 1/a_1^2 & 0 & 0 \\ a_3^2 b_3 & b_3 & 1/a_1 & -a_1 a_2 \\ -a_1^2 b_3 & 0 & 0 & a_1^2 \end{bmatrix}
\] (3.2)

with \( a_1 \neq 0 \) and arbitrary \( a_2, b_3 \). Again we note the determinant is 1 so this 3-parameter group lies in \( SO^2_2 \). Note \( \dim O(N_4) = 3 \) while from Example 3.6 we have \( \dim \hat{O}(N_4) = 6 \).

We consider three 1-parameter subgroups.

\[
\begin{align*}
a_2 = b_3 &= 0 & \text{I} \\
a_1 = 1, & b_3 = 0 & \text{II} \\
a_1 = 1, & a_2 = 0 & \text{III}
\end{align*}
\]

We make a \( 2 + 2 \) decomposition of our space into \( [v_1, u_1] \oplus [v_2, u_2] \). If we think of \( a_1 = \pm e^t \), then subgroup I is a boost by \( t \), respectively \( 2t \), on the
two subspaces. Subgroups II and III are both horocyclic translations on the
two subspaces, and they commute with each other.

Again, we can see what the part of $\widetilde{O}$ outside $O$ looks like from an
Iwasawa decomposition $\widetilde{O} = KAN \cong O_2^2$. Clearly, $I$ is a subgroup of $A$
while II and III are subgroups of $N$. Now $O_2^2$ has rank 2 so $\dim A = 2$.
Since $\dim K = 2$, then $\dim N = 2$ also. Thus all the rotations and half of
the boosts are the part of $\widetilde{O}$ outside $O$.

Many of our flat groups from Theorem 2.7 have such a subgroup isometrically embedded, as in fact do many others which are not flat.

**Proposition 3.13** Assume $[\mathfrak{Y}, \mathfrak{Y}] \subseteq \mathfrak{U}$ and $\dim \mathfrak{U} = \dim \mathfrak{Y} \geq 2$. Let
$u_1, u_2 \in \mathfrak{U}$ and $v_1, v_2 \in \mathfrak{Y}$ be such that $\langle u_i, v_i \rangle = 1$. If, on $[v_1, v_2]$, $j(u_1)^2 = -I$ and $j(u) = 0$ for $u \notin [u_1]$, then $[u_1, u_2, v_1, v_2] \cong \mathfrak{h}_3 \times \mathbb{R}$.

**Proof:** We may assume without loss of generality that $j(u_1)v_1 = v_2$. Then
a straightforward calculation shows that $[v_1, v_2] = u_1$ and the result follows.

**Corollary 3.14** For any $N$ containing such a subgroup,

$I_{\text{qi}}(N) < I_{\text{aut}}(N) < I(N)$.  

**Example 3.15** Next we consider some $N_4$ with only partially degenerate
center. Here we have the basis $\{v, e, z, u\}$ of $n_4$ with structure equation
$[v, e] = z$ and signature $(+ \bar{\varepsilon} \varepsilon -)$. We take $f$ again as above, and the metric
tensor

$$
\eta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \bar{\varepsilon} & 0 \\
0 & \varepsilon & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
$$

Applying Solve to $f^T \eta f = \eta$ directly is quite wasteful this time, producing
many solutions with $\varepsilon = 0$ which must be discarded, and producing dupli-
cates of the 4 correct solutions. After removing the rubbish, however, it is
easy to check that there are no more again. Thus we obtain

\[
f = \begin{bmatrix}
\pm 1 & 0 & 0 & 0 \\
 a_2 & \pm 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 \mp \bar{\epsilon} a_2^2/2 & \mp \bar{\epsilon} a_2 & 0 & \pm 1
\end{bmatrix}
\]

or

\[
f = \begin{bmatrix}
\pm 1 & 0 & 0 & 0 \\
 a_2 & \mp 1 & 0 & 0 \\
 0 & 0 & -1 & 0 \\
 \mp \bar{\epsilon} a_2^2/2 & \mp \bar{\epsilon} a_2 & 0 & \pm 1
\end{bmatrix}
\]

(3.3)

with arbitrary \(a_2\). Now we have the determinant \(\pm 1\) for these 1-parameter subgroups, and both are contained in \(O_{3+}^1, O_{2+}^2, \) or \(O_{1+}^3\), according to the choices for \(\bar{\epsilon}\) and \(\epsilon\), respectively \(\bar{\epsilon} = \epsilon = 1, \bar{\epsilon} \neq \epsilon, \) or \(\bar{\epsilon} = \epsilon = -1\). Again, these are all horocyclic translations as found in [3], modulo change of basis.

This yields \(\dim O(N_4) = 1\), while in the next example we shall see that \(\dim O(N_4) = 2\).

Since \(N\) is complete, so are all Killing fields. It follows [16, p. 255] that the set of all Killing fields on \(N\) is the Lie algebra of the (full) isometry group \(I(N)\). Also, the 1-parameter groups of isometries constituting the flow of any Killing field are global. Thus integration of a Killing field produces (global, not just local) isometries of \(N\).

**Example 3.16** Consider again the unique [6], nonabelian, 2-step nilpotent Lie algebra \(n_4\) of dimension 4. We take a metric tensor so that the center is only partially degenerate, as in the preceding Example. Thus we choose the basis \(\{v, e, z, u\}\) with structure equation \([v, e] = z\) and nontrivial inner products

\[
\langle v, u \rangle = 1, \quad \langle e, e \rangle = \bar{\epsilon}, \quad \langle z, z \rangle = \epsilon.
\]

This basis is partly null and partly orthonormal. The associated simply connected group \(N_4\) is isomorphic to \(H_3 \times \mathbb{R}\) and we realize it as the real matrices

\[
\begin{bmatrix}
1 & x & z & 0 & 0 \\
0 & 1 & y & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & w \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(3.4)
so this group is diffeomorphic to $\mathbb{R}^4$. Take
\[ v = \frac{\partial}{\partial x}, \quad e = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad z = \frac{\partial}{\partial z}, \quad u = \frac{\partial}{\partial w}, \]
which realizes the structure equation in the given representation. (The double use of $z$ should not cause any confusion in context.) The nontrivial adjoint maps are
\[ \text{ad}^1_v z = \varepsilon \bar{\varepsilon} e \quad \text{and} \quad \text{ad}^1_\varepsilon z = -\varepsilon u \]
so the nontrivial covariant derivatives are
\[ \nabla_v e = -\nabla_e v = \frac{1}{2} z, \]
\[ \nabla_z v = \nabla_v z = -\frac{1}{2} \varepsilon \bar{\varepsilon} e, \]
\[ \nabla_\varepsilon e = \nabla_\varepsilon z = \frac{1}{2} \varepsilon u. \]

**Remark 3.17** The space $N_4$ is not flat; in fact, a direct computation shows that
\[ \langle R(z,v)v, z \rangle = \frac{1}{4} \bar{\varepsilon} \quad \text{and} \quad \langle R(v,e)e, v \rangle = -\frac{3}{4} \varepsilon. \]

We consider Killing fields $X$ and write $X = f_1 u + f_2 z + f_3 v + f_4 e$. We denote partial derivatives by subscripts. From Killing’s equation we obtain the system.
\[ f_1^1 = f_2^2 = f_3^3 = 0 \quad (3.6) \]
\[ f_1^1 + f_2^2 = 0 \quad (3.7) \]
\[ \varepsilon f_2^1 + f_3^3 = 0 \quad (3.8) \]
\[ f_4^1 + x f_4^3 = 0 \quad (3.9) \]
\[ \varepsilon f_4^1 + f_2^2 = 0 \quad (3.10) \]
\[ \varepsilon f_4^1 + f_3^3 = 0 \quad (3.11) \]
\[ \varepsilon f_4^1 + x f_3^3 = 0 \quad (3.12) \]
\[ \varepsilon \bar{\varepsilon} f_4^1 + f_2^2 = f_3^3 \quad (3.13) \]

Simple considerations of second derivatives yield these relations.
\[ f_1^1 = f_{ww}^1 = f_{wy}^1 = f_{wz}^1 = 0 \]
\[ f_2^2 = f_{ww}^2 = f_{wx}^2 = 0 \]
\[ f_3^3 = f_{xx}^3 = f_{xz}^3 = f_{zz}^3 = 0 \]
\[ f_4^4 = f_{xx}^4 = f_{xz}^4 = 0 \]

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From the first we get \( f_w^1 = A \), and then from (3.7) \( f^3_x = -A \), so in particular \( f_{xy}^3 = 0 \). Then (3.12) and \( f_w^4 = f_{xy}^3 = f_{xz}^3 = 0 \) imply \( f^3_y = 0 \) and therefore \( f_{yz}^3 = 0 \) also. Hence,

\[
f^3 = -Ax + C_1. 
\]

Also, (3.8) and \( f^3_y = 0 \) imply \( f_{wy}^2 = 0 \), and (3.13) and \( f_{zx}^4 = 0 \) imply \( f_{yx}^2 = f_{xy}^3 = f_{xz}^3 = 0 \) imply \( f^3_z = f^3_y = 0 \). It then follows that

\[
\begin{align*}
f^1 &= Aw + F^1(y, z), \\
 f^2 &= F^2(x, y), \quad \text{with } F^2_{xy} = -A, \\
 f^3 &= -Ax + C_1, \\
 f^4 &= f^4(x, y, z), \quad \text{with } f^4_{xz} = 0 \end{align*}
\]

Now, using (3.11) and \( f^4_{xz} = 0 \), after a few steps we obtain \( F^1_z = C_2 \) whence

\[
F^1(y, z) = C_2 z + G^1(y). 
\]

Then (3.10) gives us \( f^4_z = -\varepsilon f^4_{xz} = 0 \) so

\[
f^4 = -\varepsilon C_2 - F^2_x(x, y). 
\]

From (3.9) and \( f^4_z = 0 \) we get \( 0 = f^4_y = -F^2_{xy} = A \) whence

\[
A = 0. 
\]

From (3.13) and \( f^4_z = 0 \) we get \( f^4_y = F^2_y = C_1 \) whence \( F^2(x, y) = C_1 y + G^2(x) \); hence

\[
\begin{align*}
f^1 &= C_2 z + G^1(y), \\
 f^2 &= C_1 y + G^2(x), \\
 f^3 &= C_1, \\
 f^4 &= -\varepsilon C_2 - G^2_x(x). 
\end{align*}
\]

Finally, (3.11) implies \( -\varepsilon G^2_{xx} + G^1_y + C_2 x = 0 \) so \( -\varepsilon G^2_{xxx} = -C_2 \) and hence

\[
\begin{align*}
 G^2_{xx} &= -\varepsilon C_2, \\
 G^2_{xx} &= -\varepsilon C_2 x + C_3, \\
 G^2_x &= \frac{\varepsilon C_2}{2} x^2 + C_3 x + C_4, \\
 G^2 &= \frac{\varepsilon C_2}{6} x^3 + \frac{C_3}{2} x^2 + C_4 x + C_5. 
\end{align*}
\]

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Then $G^1_y = \bar{\varepsilon}G^2_{xx} - C_2 x = \bar{\varepsilon}C_3$ whence

$$G^1(y) = \bar{\varepsilon}C_3 y + C_6.$$ and we are done.

$$f^1 = \bar{\varepsilon}C_3 y + C_2 z + C_6$$

$$f^2 = \frac{\bar{\varepsilon}C_2}{6} x^3 + \frac{C_3}{2} x^2 + C_4 x + C_1 y + C_5$$

$$f^3 = C_1$$

$$f^4 = -\bar{\varepsilon}C_2 \frac{x^2}{2} - C_3 x - (\varepsilon C_2 + C_4)$$

Therefore, dim $I(N_4) = 6$.

The Killing vector fields with $X(0) = 0$ arise for $C_1 = C_5 = C_6 = 0$ and $C_4 = -\varepsilon C_2$, so their coefficients are

$$f^1 = \bar{\varepsilon}C_3 y + C_2 z,$$

$$f^2 = \frac{\bar{\varepsilon}C_2}{6} x^3 + \frac{C_3}{2} x^2 - \varepsilon C_2 x,$$

$$f^3 = 0,$$

$$f^4 = -\bar{\varepsilon}C_2 \frac{x^2}{2} - C_3 x.$$ Therefore dim $\tilde{O}(N_4) = 2$.

**Proposition 3.18** The Killing vector fields $X$ with $X(0) = 0$ are

$$X = -\left(\frac{\bar{\varepsilon}C_2}{2} x^2 + C_3 x\right) \frac{\partial}{\partial y} - \left(\frac{\bar{\varepsilon}C_2}{3} x^3 + \frac{C_3}{2} x^2 + \varepsilon C_2 x\right) \frac{\partial}{\partial z}$$

$$+ (\varepsilon C_3 y + C_2 z) \frac{\partial}{\partial w}.$$ Their integral curves are the solutions of

$$\dot{x}(t) = 0,$$

$$\dot{y}(t) = -\frac{\bar{\varepsilon}C_2}{2} x^2 - C_3 x,$$

$$\dot{z}(t) = -\frac{\bar{\varepsilon}C_2}{3} x^3 - \frac{C_3}{2} x^2 - \varepsilon C_2 x,$$

$$\dot{w}(t) = \varepsilon C_3 y + C_2 z.$$
and the solutions with initial condition \((x_0, y_0, z_0, w_0)\) are

\[
\begin{align*}
x(t) &= x_0, \\
y(t) &= -\left(\frac{\varepsilon C_2}{2} x_0^2 + C_3 x_0\right) t + y_0, \\
z(t) &= -\left(\frac{\varepsilon C_2}{3} x_0^3 + \frac{C_3}{2} x_0^2 + \varepsilon C_2 x_0\right) t + z_0, \\
w(t) &= -\left(\frac{\varepsilon C_2^2}{3} x_0^3 + C_2 C_3 x_0^2 + (\varepsilon C_2^2 + \varepsilon C_3^2) x_0\right) \frac{t^2}{2} + (\varepsilon C_3 y_0 + C_2 z_0) t + w_0. \\
\end{align*}
\]

Thus the 1-parameter group \(\Phi_t\) of isometries is given by

\[
\begin{align*}
\Phi_t^1 &= x, \\
\Phi_t^2 &= -\left(\frac{\varepsilon C_2}{2} x^2 + C_3 x\right) t + y, \\
\Phi_t^3 &= -\left(\frac{\varepsilon C_2}{3} x^3 + \frac{C_3}{2} x^2 + \varepsilon C_2 x\right) t + z, \\
\Phi_t^4 &= -\left(\frac{\varepsilon C_2^2}{3} x^3 + C_2 C_3 x^2 + (\varepsilon C_2^2 + \varepsilon C_3^2) x\right) \frac{t^2}{2} + (\varepsilon C_3 y + C_2 z) t + w, \\
\end{align*}
\]

and its Jacobian is

\[
\Phi_{ts} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-\frac{\varepsilon C_2}{2} x + C_3 & 1 & 0 & 0 \\
-\frac{\varepsilon C_2^2}{3} x^2 + C_3 x + \varepsilon C_2 & 0 & 1 & 0 \\
-\frac{\varepsilon C_2^2}{3} x^3 + 2 C_2 C_3 x + (\varepsilon C_2^2 + \varepsilon C_3^2) t \frac{t^2}{2} & \varepsilon C_3 & C_2 t & 1
\end{bmatrix}.
\]

Hence

\[
\begin{align*}
\Phi_{ts} v &= v - (C_3 + \varepsilon C_2 x) t e - \varepsilon C_2 t z - (\varepsilon C_2^2 x^2 + 2 C_2 C_3 x + \varepsilon C_2^2 + \varepsilon C_3^2) \frac{t^2}{2} u, \\
\Phi_{ts} e &= e + (\varepsilon C_3 + C_2 x) t u, \\
\Phi_{ts} z &= z + C_2 t u, \\
\Phi_{ts} u &= u.
\end{align*}
\]

Evaluating at the identity element of \(N_4\), \((0, 0, 0, 0) \in \mathbb{R}^4\),

\[
\begin{align*}
\Phi_{ts} v &= v - C_3 t e - \varepsilon C_2 t z - \frac{\varepsilon C_2^2 + \varepsilon C_3^2}{2} t^2 u, \\
\Phi_{ts} e &= e + \varepsilon C_3 t u, \\
\Phi_{ts} z &= z + C_2 t u, \\
\Phi_{ts} u &= u.
\end{align*}
\]
Note that $C_2 = 0$ corresponds to (3.3). On the basis $\{v, e, z, u\}$, the matrix of such a $\Phi_t$ (at the identity) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-C_3 t & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{1}{2} \varepsilon C_3^2 t^2 & \varepsilon C_3 t & 0 & 1
\end{pmatrix}.
\]

**Proposition 3.19** $\Phi_t \in I^{spl}(N_4)$ if and only if $C_2 = C_3 = 0$. $\Phi_t \in I^{ad}(N_4)$ if and only if $C_2 = 0$. Thus $I^{spl}(N_4) < I^{ad}(N_4) < I(N_4)$. □

Finally, we give the matrix for a $\Phi_t$ (at the identity) on the basis $\{v, e, z, u\}$, as in (3.3) but now with $C_3 = 0$ instead. We find

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\varepsilon C_2 t & 0 & 1 & 0 \\
-\frac{1}{2} \varepsilon C_2^2 t^2 & 0 & C_2 t & 1
\end{pmatrix}.
\]

And yet again, these are horocyclic translations as found in [3], modulo change of basis.

**Acknowledgments**

Once again, Parker wishes to thank the Departamento at Santiago for its fine hospitality.

**References**


