1.7 - The Precise Definition of a Limit

We return to the idea of "bounding" the limit of interest inside some interval.

Let us consider a motivating ex:

\[ f(x) = \begin{cases} 
 2x + 1 & \text{if } x \neq 3 \\
 4 & \text{if } x = 3 
\end{cases} \]

We want to answer this question by knowing \( \lim_{x \to 3} f(x) = 5 \).

We have used both estimates and limit laws to answer this question in the past. Today, we consider a new question, namely:

How close to 3 does \( x \) need to be so that \( f(x) \) differs by less than 0.1?

To start to answer this, we notice that we are looking at distances on the x-axis, we have

\[ 1 \quad 2 \quad 3 \]

Notice these x-values are both the same distance from 3, so we can just denote both of these as \( 2 - \epsilon \) and \( 3 + \epsilon \).

\[ 0 \quad 1 \quad 2 - \epsilon \quad 3 + \epsilon \]

We with \( f(x) \), on the y-axis, we have that...
Noting that both of these possibilities can be represented by the expression $|f(x) - 5|$, we are saying $|f(x) - 5| < 0.1$.

For what $x$-values is this true?

We know limits are not always evaluated at $a$, and in this case it is not (because $f(3) = 10$, which is not the limit). Just like above, this $x$-value will be within some distance of 3. How close? We'll call this number $\delta$ and write a similar equation:

$$|x - 3| < \delta.$$ 

So how question is the following: What is the number $\delta$ such that

$$|f(x) - 55| < 0.1$$ if $0 < |x - 3| < \delta$.

The 0 condition ensures we do not evaluate at 3.

Notice that if we use the equation $|f(x) - 5|$, we can have $|f(x) - 5| = |2x - 6 - 5| = |2x - 11| = 2|x - 3| < 0.1$ since $|f(x) - 5| < 0.1$ by assumption, and $|f(x) - 5| = 2|x - 3|$ by our algebra, we must have $2|x - 3| < 0.1$. Dividing by 2 gives

$$|x - 3| < 0.05,$$ but this is our answer!
So we have that \(|f(x) - 5| < 0.1\) if \(0 < |x-3| < 0.05\). Notice that to answer this question, the \(x \neq 0\) only came into play when we were dividing by 2 at the very end.

Finding our \(\delta\) would always have this structure, no matter how close we wanted \(f(x)\) to be to 5. This means if we substitute 0.1 for \(\varepsilon\) for any letter representing an arbitrary tolerance (we will use \(\varepsilon\); pronounced epsilon), we may write that, in general, for the function

\[
f(x) = \begin{cases} 2x-1 & x \neq 3 \\ x+3 & x = 3 \end{cases}
\]

the

\[
|f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x-3| < \delta = \frac{\varepsilon}{2}.
\]

In other words;

if \(3-\delta < x < 3 + \delta \ (x+3)\), then \(5-\varepsilon < f(x) < 5+\varepsilon\).

Pictorially

\[
\begin{array}{c}
\text{Graph of } f(x) = \begin{cases} 2x-1 & x \neq 3 \\ x+3 & x = 3 \end{cases} \\
\text{with } \varepsilon = 0.1, \delta = 0.05
\end{array}
\]
**DEF** Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly at \( a \) itself. Then we say the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), and we write

\[
\lim_{{x \to a}} f(x) = L
\]

if for every number \( \varepsilon > 0 \), there exists a number \( \delta > 0 \) such that if

\[0 < |x-a| < \delta, \text{ then } |f(x) - L| < \varepsilon.\]

Pictorially, this is saying you can always put a box of this type around your point of interest. This is why we said infinite limits do not technically exist: one could argue that you can't find an tolerance around \( x \) so that \( f(x) \) is sufficiently close to \( x \), but the problem is even more fundamental; it doesn't even make sense to ask the question, for example: when is \( f(x) \) within 0.1 from \( \infty \)? What would \( \infty + 0.1 \) even look like? It doesn't make sense.

**Ex.** For \( \lim_{{x \to 2}} (x^3 - 3x + 4) = 6 \), find values of \( \delta \) which correspond to \( \varepsilon = 0.2 \) when \( \varepsilon = 0.00 \).

**Soln.** We need a \( \delta \) such that when \(|x-2| < \delta\), \(|f(x) - 6| < 0.2\). We notice that \(|f(x) - 6| < 0.2\) means \(-0.2 < x^3 - 3x + 4 - 6 < 0.2\).

Graphing this (on a calculator), we get
for $5.8 < x^3 - 3x + 4 < 6.2$. We see that $f(x) = 6.2$ when $x \approx 2.002$ and $f(5) = 5.8$ when $x \approx 1.9774$, so $2 - 8 < 1.9774$, giving $8 < 0.0226$, and for $x = 2.022$, $2 + 8 < 2.0214$. We take the smaller $S$ (since both are true when $S$ is smaller) and have $S = 0.0219$.

More generally, if $|f(a + S) - f(a)| < \epsilon$, we have that $\left| (2 + S)^3 - 3(2 + S) + 4 - 16 \right| = \left| 8^3 + 16S^2 + 9S \right| \\
\leq |8^3| + |16S^2| + |9S|$,.

Now assume $S < 1$ (this is reasonable). Then $|8^3| + |16S^2| + |9S| \leq 9S + 9S + 9S \leq 27S$, so we can set $S = \frac{\epsilon}{27}$ for any $\epsilon$.

Exercise: Prove $\lim_{x \to 3} x^2 = 9$.

Let $\epsilon > 0$. We have to find $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left| x^2 - 9 \right| < \epsilon$.

Notice $\left| x^2 - 9 \right| = \left| x + 3 \right| \left| x - 3 \right| < \epsilon$. If we can find a positive constant $C$ such that $\left| x + 3 \right| < C$, we can substitute in here. Again, it is reasonable to assume $\delta < 1$. So $\left| x - 3 \right| < \delta$ gives $0 < x - 3 \delta < 1$ and $2 < x < 4$. Then $5 < x + 3 < 7$. We pick the bigger value and divide $\epsilon$, so $S = \frac{\epsilon}{7}$ will get us in the range we want and we have $\left| x - 3 \right| < \frac{\epsilon}{7}$.

Proof: Let $\delta \leq \frac{\epsilon}{7}$, and $S = \min \left\{ 1, \frac{\epsilon}{7} \right\}$. If $0 < |x - 3| < \delta$, then $2 < x < 4$, so $|x + 3| < 7$. Since $|x - 3| < \frac{\epsilon}{7}$, then
\[ |x^2 - 9| = |x+3||x-3| < 7 \cdot \frac{3}{x-4} = 3. \text{ So the limit } x^2 = 9. \]

To illustrate limit laws don't just fall out of the sky, let's prove the sum law:

**Ex.** Prove that if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \), then \( \lim_{x \to a} (f(x) + g(x)) = L + M \).

**Proof:** Let \( \varepsilon > 0 \) be given. We must find a \( \delta > 0 \) such that if \( 0 < |x-a| < \delta \), then \( |f(x) + g(x) - (L + M)| < \varepsilon \).

By the triangle inequality,

\[ |f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|. \]

Let \( \varepsilon/2 > 0 \) (because \( \varepsilon \) is), \( \exists \delta_1, \delta_2 > 0 \) such that if \( 0 < |x-a| < \delta_1 \), then \( |f(x) - L| < \varepsilon/2 \). (Same for \( g(x) \)).

Like before we have a \( \delta_1, \delta_2 \), so we pick the min, \( \delta = \min \delta_1, \delta_2 \). So that if \( 0 < |x-a| < \delta \),

if \( 0 < |x-a| < \delta \), then \( 0 < |x-a| < \delta_1 \) and \( 0 < |x-a| < \delta_2 \), so \( |f(x) - L| < \varepsilon/2 \) and \( |g(x) - M| < \varepsilon/2 \).

Finally we have

\[ |f(x) + g(x) - (L + M)| < |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]