1.4 - The Tangent and Velocity Problem

Tangent lines: tangent is Latin for "touching".

If we try to generalize from Euclidean (high school) geometry to curves of any type, we'd say that a tangent line is any line that intersects a curve at most once.

Is this enough?

Consider the following curve.

We can definitely draw a tangent line at $p$. But once we extend it, it intersects the line twice. So our previous definition was insufficient.

We could draw a line through $p$ that only intersects once, but it wouldn't be tangent.
Let's start with an example we know well:  
\[ y = x^2 \]

We will try to find the tangent line at \((1,1)\).

To write an equation, we need a slope. That requires two lines.

We can draw a secant line (intersects a curve more than once, Latin for "cutting") that goes through \( p \) to help us estimate. Let's pick \((2,4)\).

Then our slope is \( m = \frac{4-1}{2-1} = 3 \).

Could we get a better estimate if we were closer?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \text{MPa} )</th>
<th>( \text{MPa} )</th>
<th>( \text{MPa} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>2.25</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>1.21</td>
<td>2.1</td>
<td></td>
</tr>
<tr>
<td>1.01</td>
<td>1.0201</td>
<td>2.01</td>
<td></td>
</tr>
<tr>
<td>1.001</td>
<td>1.002001</td>
<td>2.001</td>
<td></td>
</tr>
<tr>
<td>0.999</td>
<td>0.98001</td>
<td>1.999</td>
<td></td>
</tr>
<tr>
<td>0.9999</td>
<td>0.9980001</td>
<td>1.9999</td>
<td></td>
</tr>
</tbody>
</table>

Average number:  \( \frac{2.5 + 1.5}{2} = \frac{4}{2} = 2 \).
As we get closer and closer near to \((1,1)\) from the left and the right, we "converge" to the same slope: \(m = 2\).

Note: We have just found our first limit.

We need our tangent line to go through the point \((1,1)\), so from here we just use point-slope:

\[
y - 1 = 2(x - 1) = 2x - 2
\]

\[
y = 2x - 1.
\]

Ex) Suppose a ball is dropped 450m off the ground. Find the velocity of the ball at \(t = 5\) sec.

The distance the ball has traveled over a time \(t\):

\[
S(t) = 4.9t^2.
\]

We can compute close to 5:

\[
\Delta v = \frac{\Delta \text{position}}{\Delta \text{time}}
\]

We take closer and closer approximations:

<table>
<thead>
<tr>
<th>Time</th>
<th>Average Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5 \leq t \leq 6)</td>
<td>53.9</td>
</tr>
<tr>
<td>(5.0 \leq t \leq 5.1)</td>
<td>49.99</td>
</tr>
<tr>
<td>(5 \leq t &lt; 5.05)</td>
<td>49.245</td>
</tr>
<tr>
<td>(5 \leq t &lt; 5.01)</td>
<td>49.049</td>
</tr>
<tr>
<td>(5 \leq t &lt; 5.007)</td>
<td>49.0049</td>
</tr>
</tbody>
</table>
The average velocity is 49 m/s, so we have our answer.

Notice: other than plugging in a point, this is what we did before.

- We took a limit.

Meaning: To solve our tangent line problem, we need limits. Tangent lines are the limit of the slopes of the secant lines.

We took the limit in both of the above examples.

Idea is convergence: they converge to the lines/values found.
In our first example, we computed
\[ \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2. \]

In the second example, we computed
\[ \lim_{t \to 5} \frac{4.9t^2 - 122.5}{t - 5} = 49. \]

In college algebra, finding asymptotes was equivalent to finding a limit (rational functions).
In geometry, \( \infty \) as the number of sides of a polygon \( \to \), it approaches a circle.
**DEF** The intuitive definition of a limit.

Suppose \( f(x) \) is defined when \( x \) is near the number \( a \) (this means the function \( f \) is defined on some open interval near \( a \), except possibly \( a \) itself). Then we write

\[
\lim_{{x \to a}} f(x) = L
\]

and say "the limit of \( f(x) \), as \( x \) approaches \( a \), is \( L \)," if we can make the values of \( f(x) \) arbitrarily close to \( L \) (as close as we'd like) by restricting \( x \) to be sufficiently close to \( a \) (on either side of \( a \)) but not equal to \( a \).

Note: Not every function has a limit at \( a \).

**Ex1 Step functions:**

If 1 approach \( x = 1 \) from the left, I get that the limit is 1. But from the right, it is 2.