Algebraic Functions

**DEF:** An algebraic function \( f \) is a function which can be constructed using algebraic operations (addition, subtraction, multiplication, division, taking square roots), starting with polynomials.

**Fact:** All rational functions are algebraic (but not all algebraic functions are rational).

Unphysical: These functions have very different shapes.

**Ex1:** \( f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 25 + \sqrt{x}} - (x+2)^2 \sqrt[3]{x} + 1 \) is an algebraic function.

**Ex:** In physics, we use an algebraic function to estimate the mass of a particle traveling with some velocity \( v \):

\[
m = f(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\( c = \) speed of light. \( m_0 = \) mass of particle at rest \( (v = 0) \).
Trigonometric Functions

In Calculus, we use measurements of radians unless it is specified to use degrees.

The graphs of \( f(x) = \sin(x) \) and \( g(x) = \cos(x) \) are:

The domain for both is \((-\infty, \infty)\). The range is \([-1, 1]\).

The zeros of \( \sin(x) \) occur at \( x = n\pi \), \( n \) an integer.

The zeros of \( \cos(x) \) occur at \( \frac{(2n+1)\pi}{2} \), \( n \) an integer.

These functions are periodic with a period of \( 2\pi \). This means the following identities hold:

1. \( \sin(x + 2\pi) = \sin(x) \)
2. \( \cos(x + 2\pi) = \cos(x) \).
Ex: Find the domain of \( f(x) = \frac{1}{1 - 2\cos(x)} \).

**Solu:** We look at the denominator:

\[
1 - 2\cos(x) = 0 \\
\cos(x) = \frac{1}{2} \quad \text{This is true when } x = \frac{\pi}{3}.
\]

But by our identity relation: \( \cos(\frac{\pi}{3}) = \cos(\frac{\pi}{3} + 2\pi) \), so the domain is all real numbers except when \( x = \frac{\pi}{3} + 2\pi n \) where \( n \) is an integer.

The tangent function, \( f(x) = \tan(x) \), is defined as a quotient of sine and cosine:

\[
\tan(x) = \frac{\sin(x)}{\cos(x)}.
\]

While the range of \( \tan(x) \) is \( (-\infty, \infty) \), it is undefined wherever \( \cos(x) = 0 \), so for \( x = \frac{\pi}{2} + 2\pi n \). The graph:

Note: The period is \( \pi \).

So our identity is \( \tan(x + \pi) = \tan(x) \) for all \( x \).
Exponential Functions

Definition: An exponential function $f$ is of the form $f(x) = b^x$, where the base $b$ is a positive constant.

Fact: The domain of an exponential function is $(-\infty, \infty)$. The range is $(0, \infty)$.

If $b < 1$, the graph has the general shape:

If $b > 1$, then the graph has the general shape:
Logarithmic Functions

**DEF**: A logarithmic function is defined to be the inverse of the exponential function, expressed as $f(x) = \log_b x$, where $b$ is a positive constant.

**Graph**: Domain: $(0, \infty)$, Range: $(-\infty, \infty)$.

As the base decreases, the height of the graph increases.

Fact: All log functions have $\log_b(1) = 0$. 
1.3 - New Functions from Old Ones

Using the functions described in the previous sections, we can create new functions using:
1. Translations (shifting)
2. Vertical/Horizontal Stretching
3. Reflecting
4. Composing

**Vertical/Horizontal Shifts (Translations)**

- Suppose \( f(x) \). To obtain the graph of:
  1. \( y = f(x) + c \), shift the graph \( y = f(x) \) \( c \) units up
  2. \( y = f(x) - c \), shift graph \( c \) units down
  3. \( y = f(x - c) \), shift \( c \) units to the right
  4. \( y = f(x + c) \), shift \( c \) units to the left

**Stretching/Reflecting (c > 1)**

- \( y = cf(x) \), stretch vertically by a factor of \( c \)
- \( y = f(\frac{x}{c}) \), shrink vertically by a factor of \( c \)
- \( y = f(cx) \), shrink horizontally by factor of \( c \)
- \( y = f(\frac{x}{c}) \), shrink horizontally by factor of \( c \)
- \( y = -f(x) \), reflect about the \( x \)-axis
- \( y = f(-x) \), reflect about the \( y \)-axis
Ex) $y = x^2 - 4x + 5 = x^2 - 4x + 4 + 1$
$= (x - 2)^2 + 1$
Take $x^2 = f(x)$. Shift to the right 2, up 1.

Ex) $y = \frac{1}{4} \tan \left( x - \frac{\pi}{4} \right)$
- Shift $\tan(x)$ to the right by $\frac{\pi}{4}$.
- Shrink vertically by a factor of 4.

Composition

DEF: Two functions $f$ and $g$ have their composite defined by $(f \circ g)(x) = f(g(x))$.
Domain is that shared by both $f$ and $g$.

Ex) If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find $f \circ g$ and the domain.

\[ f \circ g(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt{2-x}. \]
$x \leq 2$, so domain is $(-\infty, 2]$.

Ex) Find $f \circ g \circ h$ for $f(x) = 3x - 2$, $g(x) = \sin(x)$, $h(x) = x^2$

$= f(g(x^2)) = f(\sin(x^2)) = 3\sin(x^2) - 2$. 