L18 χ^2-distributions in ANOVA tables

1. χ^2-distributions in ANOVA table I

   (1) χ^2-distributions
   Recall: $X \sim N(\mu, \Sigma)$, $A\Sigma A = A = A' \Longrightarrow X'AX \sim \chi^2(\mu' A\mu, \text{tr}(A\Sigma))$.

   (2) ANOVA table for M: $Y \sim N(X\beta, \sigma^2\Sigma)$ where $\mathcal{R}(1) \subset \mathcal{R}(X)$ and $\text{rank}(X) = r$.

Source	SS	DF
Model	SSM= $(\Sigma^{-1/2}Y)'\left[(\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+ - (\Sigma^{-1/2}1)(\Sigma^{-1/2}1)^+\right](\Sigma^{-1/2}Y)$	$r-1$
Error	SSE= $(\Sigma^{-1/2}Y)'\left[I - (\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+\right](\Sigma^{-1/2}Y)$	$n-r$
C.Total	C.SSTO= $(\Sigma^{-1/2}Y)'\left[I - (\Sigma^{-1/2}1)(\Sigma^{-1/2}1)^+\right](\Sigma^{-1/2}Y)$	$n-1$

   (3) χ^2-distributions in ANOVA table I
   Note that there is H_0 under which model M is reduced to $M_1 : Y \sim N(1_n\gamma, \sigma^2\Sigma)$.
   (i) $\frac{\text{SSE}}{\sigma^2} \sim \chi^2(n-r)$ (ii) $\frac{\text{SSM}}{\sigma^2} \overset{H_0}{\sim} \chi^2(r-1)$ (iii) $\frac{\text{C.SSTO}}{n-1} \overset{H_0}{\sim} \chi^2(n-1)$

   Proof. (i) $Y \sim N(X\beta, \sigma^2\Sigma) \Longrightarrow \Sigma^{-1/2}Y \sim N(\Sigma^{-1/2}X\beta, \sigma^2 I_n)$.
   $\frac{\text{SSE}}{\sigma^2} = (\Sigma^{-1/2}Y)'A(\Sigma^{-1/2}Y)$ where $A = \frac{I-(\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+}{\sigma^2}$.
   But $A\sigma^2 IA = A = A'$, $(\Sigma^{-1/2}X\beta)'A(\Sigma^{-1/2}X\beta) = 0$ and $\text{tr}(A\sigma^2 I) = n - r$.
   So $\frac{\text{SSE}}{\sigma^2} \sim \chi^2(n-r)$.

   (ii) Under H_0, $Y \sim N(1_n\gamma, \sigma^2\Sigma) \Longrightarrow \Sigma^{-1/2}Y \sim N(\Sigma^{-1/2}1_n\gamma, \sigma^2 I_n)$.
   $\frac{SSM}{\sigma^2} = (\Sigma^{-1/2}Y)'A(\Sigma^{-1/2}Y)$ where $A = \frac{(\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+ - (\Sigma^{-1/2}1_n)(\Sigma^{-1/2}1_n)^+}{\sigma^2}$.
   But $A\sigma^2 I_n A = A = A'$, $(\Sigma^{-1/2}1_n\gamma)'A(\Sigma^{-1/2}1_n\gamma) = 0$, $\text{tr}(A\sigma^2 I_n) = r - 1$.
   So under H_0, $\frac{\text{SSM}}{\sigma^2} \sim \chi^2(r-1)$.

   (iii) Under H_0, $Y \sim N(1_n\gamma, \sigma^2\Sigma) \Longrightarrow \Sigma^{-1/2}Y \sim N(\Sigma^{-1/2}1_n\gamma, \sigma^2 I_n)$.
   $\frac{C.SSTO}{\sigma^2} = (\Sigma^{-1/2}Y)'A(\Sigma^{-1/2}Y)$ where $A = \frac{I-(\Sigma^{-1/2}1_n)(\Sigma^{-1/2}1_n)^+}{\sigma^2}$.
   But $A\sigma^2 I_n A = A = A'$, $(\Sigma^{-1/2}1_n\gamma)'A(\Sigma^{-1/2}1_n\gamma) = 0$, $\text{tr}(A\sigma^2 I_n) = n - 1$.
   So under H_0, $\frac{\text{C.SSTO}}{\sigma^2} \sim \chi^2(n-1)$.

   Ex1: For $Y \sim N(X\beta, \sigma^2 I_n)$ with $\mathcal{R}(1_n) \subset \mathcal{R}(X)$, in ANOVA table

Source	SS	DF
Model	SSM= $Y'(XX^+ - 1_n1_n^+)Y$	$r-1$
Error	SSE= $Y'(I_n - XX^+)Y$	$n-r$
C.Total	C.SSTO= $Y'(I_n - 1_n1_n^+)Y$	$n-1$

   $\frac{\text{SSM}}{\sigma^2} \overset{H_0}{\sim} \chi^2(r-1)$, $\frac{\text{SSE}}{\sigma^2} \sim \chi^2(n-r)$ and $\frac{\text{C.SSTO}}{\sigma^2} \overset{H_0}{\sim} \chi^2(n-1)$.

2. χ^2-distributions in ANOVA table II

   (1) ANOVA table for M: $Y \sim N(X\beta, \sigma^2\Sigma)$ where $\mathcal{R}(1) \not\subset \mathcal{R}(X)$ and $\text{rank}(X) = r$.

Source	SS	DF
Model	SSM= $(\Sigma^{-1/2}Y)'(\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+(\Sigma^{-1/2}Y)$	r
Error	SSE= $(\Sigma^{-1/2}Y)'\left[I - (\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+\right](\Sigma^{-1/2}Y)$	$n-r$
U.Total	U.SSTO= $(\Sigma^{-1/2}Y)'I_n(\Sigma^{-1/2}Y)$	n

   (2) χ^2-distributions in ANOVA table II
   Note that under $H_0 : \beta = 0$ model M is reduced to $M_0 : Y \sim N(0, \sigma^2\Sigma)$.
   (i) $\frac{\text{SSE}}{\sigma} \sim \chi^2(n-r)$ (ii) $\frac{\text{SSM}}{\sigma^2} \overset{H_0}{\sim} \chi^2(r)$ (iii) $\frac{\text{U.SSTO}}{n} \overset{H_0}{\sim} \chi^2(n)$

Proof. (i) See 1 (3) (i).

(ii) Under $H_0: \beta = 0$, $Y \sim N(0, \sigma^2\Sigma) \Longrightarrow \Sigma^{-1/2}Y \sim N(0, \sigma^2 I_n)$.
$\frac{SSM}{\sigma^2} = (\Sigma^{-1/2}Y)'A(\Sigma^{-1/2}Y)$ where $A = \frac{(\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+}{\sigma^2}$.
But $A\sigma^2 I_n A = A = A'$, $0'A0 = 0$, $\text{tr}(A\sigma^2 I_n) = r$.
So under H_0, $\frac{\text{SSM}}{\sigma^2} \sim \chi^2(r)$.

(iii) Under $H_0: \beta = 0$, $Y \sim N(0, \sigma^2\Sigma) \Longrightarrow \Sigma^{-1/2}Y \sim N(0, \sigma^2 I_n)$.
$\frac{U.SSTO}{\sigma^2} = (\Sigma^{-1/2}Y)'A(\Sigma^{-1/2}Y)$ where $A = \frac{I_n}{\sigma^2}$.
But $A\sigma^2 I_n A = A = A'$, $0'A0 = 0$, $\text{tr}(A\sigma^2 I_n) = n$.
So under H_0, $\frac{\text{U.SSTO}}{\sigma^2} \sim \chi^2(n)$.

Ex2: For $Y \sim N(X\beta, \sigma^2 I_n)$ with $\mathcal{R}(1_n) \not\subset \mathcal{R}(X)$, in ANOVA table

Source	SS	DF
Model	SSM= $Y'XX^+Y$	r
Error	SSE= $Y'\left(I_n - XX^+\right)Y$	$n-r$
U.Total	U.SSTO= $Y'I_nY$	n

$\frac{\text{SSM}}{\sigma^2} \overset{H_0}{\sim} \chi^2(r)$, $\frac{\text{SSE}}{\sigma^2} \sim \chi^2(n-r)$ and $\frac{\text{U.SSTO}}{\sigma^2} \overset{H_0}{\sim} \chi^2(n)$.

3. UE for σ^2 and MS in ANOVA table

(1) MS in ANOVA table I
Define MSE $= \frac{\text{SSE}}{n-r}$, MSM $= \frac{\text{SSM}}{r-1}$, MSTO $= \frac{\text{C.SSTO}}{n-1}$. Then MSE is an UE for σ^2, MSM and MSTO are UEs for σ^2 under H_0 in Model M_1.

Proof. $E\left(\frac{\text{SSE}}{\sigma^2}\right) = E\left[\chi^2(n-r)\right] = n - r \Longrightarrow E(\text{MSE}) = \sigma^2$.

$E\left(\frac{\text{SSM}}{\sigma^2}\right) \overset{H_0}{=\!=} E\left[\chi^2(r-1)\right] = r - 1 \Longrightarrow E\left(\text{MSM}\right) \overset{H_0}{=\!=} \sigma^2$.

$E\left(\frac{\text{C.SSTO}}{\sigma^2}\right) = E\left[\chi^2(n-1)\right] = n - 1 \Longrightarrow E\left(\text{MSTO}\right) = \sigma^2$.

(2) MS in ANOVA table II
Define MSE $= \frac{\text{SSE}}{n-r}$, MSM $= \frac{\text{SSM}}{r}$, MSTO $= \frac{\text{U.SSTO}}{n}$. Then MSE is an UE for σ^2, MSM and MSTO are UEs for σ^2 under $H_0: \beta = 0$ in Model M_0.
Proof. Skipped.

(3) Updated ANOVA tables

Source	SS	DF	MS
Model	SSM	$r-1$	MSM
Error	SSE	$n-r$	MSE
C.Total	C.SSTO	$n-1$	MSTO

Source	SS	DF	MS
Model	SSM	r	MSM
Error	SSE	$n-r$	MSE
U.Total	U.SSTO	n	MSTO

L19 F-distributions in ANOVA tables

1. F-distribution in ANOVA table I

   (1) F-distributions
   Recall: $X^2 \sim \chi^2(r_1)$ and $Y^2 \sim \chi^2(r_2)$ are independent $\Longrightarrow F = \frac{X^2/r_1}{Y^2/r_2} \sim F(r_1, r_2)$.
   $X \sim N(\mu, \Sigma)$, $A' = A$, $B' = B$ and $A\Sigma B = 0 \Longrightarrow X'AX$ and $X'BX$ are independent.

   (2) ANOVA table I
   For Model M: $Y \sim N(X\beta, \sigma^2\Sigma)$ with $1_n \in \mathcal{R}(X)$ and $\text{rank}(X) = r$ there exists H_0. Under H_0 M is reduced to $M_1 : Y \sim N(1_n\gamma, \sigma^2\Sigma)$. So $\Sigma^{-1/2}Y \overset{H_0}{\sim} N(\Sigma^{-1/2}1_n\gamma, \sigma^2 I_n)$. For this model M and H_0,

Source	SS	DF	MS	F	$Pr > F$
Model	SSM	$r-1$	MSM	MSM/MSE	$P(F(r-1, n-r) > F_{ob})$
Error	SSE	$n-r$	MSE		
C.Total	C.SSTO	$n-1$			

   (3) F-distribution in Table I
   $F = \frac{\text{MSM}}{\text{MSE}} \overset{H_0}{\sim} F(r-1, n-r)$.

   Proof. $F = \frac{\text{MSM}}{\text{MSE}} = \frac{\frac{\text{SSM}}{\sigma^2}/(r-1)}{\frac{\text{SSE}}{\sigma^2}/(n-r)}$ where $\frac{\text{SSM}}{\sigma^2} \overset{H_0}{\sim} \chi^2(r-1)$ and $\frac{\text{SSE}}{\sigma^2} \sim \chi^2(n-r)$.

   To have $F \overset{H_0}{\sim} F(r-1, n-r)$ we need the independence of $\frac{\text{SSM}}{\sigma^2}$ and $\frac{\text{SSE}}{\sigma^2}$ under H_0.
   But $\frac{\text{SSM}}{\sigma^2} = (\Sigma^{-1/2}Y)'A(\Sigma^{-1/2}Y)$ where $A = \frac{(\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+ - (\Sigma^{-1/2}1_n)(\Sigma^{-1/2}1_n)^+}{\sigma^2}$
   and $\frac{\text{SSE}}{\sigma^2} = (\Sigma^{-1/2}Y)'B(\Sigma^{-1/2}Y)$ where $B = \frac{I - (\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+}{\sigma^2}$.
   Clearly, $A\sigma^2 I_n B = 0$. So $\frac{\text{SSM}}{\sigma^2}$ and $\frac{\text{SSE}}{\sigma^2}$ are independent under H_0.
   Conclusion follows.

   Comment: We now have completed ANOVA table in (2)

2. F-distribution in ANOVA table II

   (1) ANOVA table II
   Model M: $Y \sim N(X\beta, \sigma^2\Sigma)$ with $1_n \notin \mathcal{R}(X)$ and $\text{rank}(X) = r$ under $H_0 : \beta = 0$ is reduced to $M_0 : Y \sim N(0, \sigma^2\Sigma)$. So $\Sigma^{-1/2}Y \overset{H_0}{\sim} N(0, \sigma^2 I_n)$. For this Model M and $H_0 : \beta = 0$,

Source	SS	DF	MS	F	$Pr > F$
Model	SSM	r	MSM	MSM/MSE	$P(F(r, n-r) > F_{ob})$
Error	SSE	$n-r$	MSE		
U.Total	U.SSTO	n			

   (2) F=distribution in ANOVA table II
   $F = \frac{\text{MSM}}{\text{MSE}} \overset{H_0}{\sim} F(r, n-r)$.

   Proof. $F = \frac{\text{MSM}}{\text{MSE}} = \frac{\frac{\text{SSM}}{\sigma^2}/(r)}{\frac{\text{SSE}}{\sigma^2}/(n-r)}$ where $\frac{\text{SSM}}{\sigma^2} \overset{H_0}{\sim} \chi^2(r)$ and $\frac{\text{SSE}}{\sigma^2} \sim \chi^2(n-r)$.

   To have $F \overset{H_0}{\sim} F(r, n-r)$ we need the independence of $\frac{\text{SSM}}{\sigma^2}$ and $\frac{\text{SSE}}{\sigma^2}$ under H_0.

But $\frac{\text{SSM}}{\sigma^2} = (\Sigma^{-1/2}Y)'A(\Sigma^{-1/2}Y)$ where $A = \frac{(\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+}{\sigma^2}$ and $\frac{\text{SSE}}{\sigma^2} = (\Sigma^{-1/2}Y)'B(\Sigma^{-1/2}Y)$ where $B = \frac{I-(\Sigma^{-1/2}X)(\Sigma^{-1/2}X)^+}{\sigma^2}$.
Clearly, $A\sigma^2 I_n B = 0$. So $\frac{\text{SSM}}{\sigma^2}$ and $\frac{\text{SSE}}{\sigma^2}$ are independent under H_0.
Conclusion follows.

Comment: (1) gives a completed ANOVA table.

3. Computation for ANOVA tables

(1) Regression with intercept
Regression with intercept $y = \beta_0+\beta_1 x_1+\beta_2 x_2+\epsilon$ has data $Y \sim N(X\beta,\, \sigma^2 I_n)$ where $X \in R^{n\times 3}$ has rank 3, and its first column is 1_n. This model is reduced to $Y \sim N(1_n\beta_0,\, \sigma^2 I_n)$ under $H_0 : \beta_1 = 0$ and $\beta_2 = 0$. Thus the model with H_0 has ANOVA table I.

```
data a;
   infile "D:\ex.txt";
   input y x1 x2 @@;
proc reg;
   model y=x1 x2;
   run;
```

SAS output displays the ANOVA Table I.

(2) Regression without intercept
Regression without intercept $y = \beta_1 x_1 + \beta_2 x_2 + \epsilon$ has data $Y \sim N(X\beta,\, \sigma^2 I_n)$. Here $X \in R^{n\times 2}$ has full column rank, but $1_n \notin \mathcal{R}(X)$. Under $H_0 : \beta_1 = 0$ and $\beta_2 = 0$ the reduced model is $Y \sim N(0,\, \sigma^2 I_n)$. Thus the model with H_0 has ANOVA Table II

```
proc reg;
   model y=x1 x2/noint;
   run;
```

SAS output displays the ANOVA Table II.

(3) One-way ANOVA
$y = \mu_1 I_1+\mu_2 I_2+\mu_3 I_3+\epsilon$ is ANOVA model with one factor and three levels. Here I_i is the indicator for the ith level, i.e., $I_i = \begin{cases} 0 & \text{if } y \text{ is not the response to level } i \text{ of the factor} \\ 1 & \text{if } y \text{ is the response to level } i \text{ of the factor} \end{cases}$.

This model has data $Y \sim N(J\mu,\, \sigma^2 I_n)$. Here $J = \begin{pmatrix} 1_{n_1} & 0 & 0 \\ 0 & 1_{n_2} & 0 \\ 0 & 0 & 1_{n_3} \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$.

Under $H_0 : \mu_1 = \mu_2 = \mu_3$ the model is reduced to $Y \sim N(1_n\mu_1,\, \sigma^2 I_n)$. So it has ANOVA table I.

```
data a;
   infile "D:\ex3.txt";
   input y Sname $ @@;
proc anova;
   class Sname;
   model y=Sname;
   run;
```

SAS output displays the ANOVA Table I.