

## L14: Tests on $l'\beta$

### 1. Pivotal quantities

Consider model  $Y = X\beta + \epsilon \sim N(X\beta, \sigma^2\Sigma)$  where  $X \in R^{n \times p}$  has full column rank.

(1) A general pivotal quantity with  $F$ -distribution

$$\text{Recall: } F = \frac{(H\beta - H\hat{\beta})'[H(X'\Sigma^{-1}X)^{-1}H']^{-1}(H\beta - H\hat{\beta})}{q \text{MSE}} \sim F(q, n - p)$$

where  $H \in R^{q \times p}$  has full row rank,  $\hat{\beta} = (\Sigma^{-1/2}X)^+ \Sigma^{-1/2}Y = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$ ,  
 $\text{SSE} = (\Sigma^{-1/2}Y)' \left[ I - (\Sigma^{-1/2}X) (\Sigma^{-1/2}X)^+ \right] (\Sigma^{-1/2}Y)$  and  $\text{MSE} = \frac{\text{SSE}}{n-p}$ .

(2) A special case of  $H = l' \in R^{1 \times p}$

$$F = \frac{(l'\hat{\beta} - l'\beta)[l'(X'\Sigma^{-1}X)^{-1}l]^{-1}(l'\hat{\beta} - l'\beta)}{\text{MSE}} \sim F(1, n - p)$$

(3) Re-write (2) equivalently as

$$t = \frac{l'\hat{\beta} - l'\beta}{S_{l'\hat{\beta}}} \sim t(n - p)$$

$S_{l'\hat{\beta}}^2 = \text{MSE } l'(X'\Sigma^{-1}X)^{-1}l$  is the estimated variance of  $l'\hat{\beta}$ ,  $\sigma_{l'\hat{\beta}}^2 = \sigma^2 l'(X'\Sigma^{-1}X)^{-1}l$ .

**Proof.** Note that  $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y \sim N(\beta, \sigma^2(X'\Sigma^{-1}X)^{-1})$ .

So  $l'\hat{\beta} \sim N(l'\beta, \sigma_{l'\hat{\beta}}^2)$  where  $\sigma_{l'\hat{\beta}}^2 = \sigma^2 l'(X'\Sigma^{-1}X)^{-1}l$  is estimated by  
 $S_{l'\hat{\beta}}^2 = \text{MSE } l'(X'\Sigma^{-1}X)^{-1}l$ .

$$\begin{aligned} & \frac{(l'\hat{\beta} - l'\beta)[l'(X'\Sigma^{-1}X)^{-1}l]^{-1}(l'\hat{\beta} - l'\beta)}{\text{MSE}} \sim F(1, n - p) \\ \iff & \left[ \frac{l'\hat{\beta} - l'\beta}{\sqrt{\text{MSE } l'(X'\Sigma^{-1}X)^{-1}l}} \right]^2 \sim [t(n - p)]^2 \iff \frac{l'\hat{\beta} - l'\beta}{S_{l'\hat{\beta}}} \sim t(n - p). \end{aligned}$$

**Ex1:** For Model  $Y = X\beta + \epsilon \sim N(X\beta, \sigma^2I)$  where  $X \in R^{n \times p}$  has full column rank,

$$F = \frac{(l'\hat{\beta} - l'\beta)[l'(X'X)^{-1}l]^{-1}(l'\hat{\beta} - l'\beta)}{\text{MSE}} \sim F(1, n - p)$$

where  $\hat{\beta} = X^+Y = (X'X)^{-1}X'Y$ ,  $\text{SSE} = Y'(I - XX^+)Y$ ,  $\text{MSE} = \frac{\text{SSE}}{n-p}$ . In

$$t = \frac{l'\hat{\beta} - l'\beta}{S_{l'\hat{\beta}}} \sim t(n - p)$$

$S_{l'\hat{\beta}}^2 = \text{MSE } l'(X'X)^{-1}l$  is the estimated variance of  $l'\hat{\beta}$ ,  $\sigma_{l'\hat{\beta}}^2 = \sigma^2 l'(X'X)^{-1}l$ .

### 2. Two-sided $t$ -tests in linear models

(1) Two equivalent tests

$H_0 : l'\beta = b$ vs $H_a : l'\beta \neq b$ Test Statistic: $F = \frac{(l'\hat{\beta} - b)[l'(X'\Sigma^{-1}X)^{-1}l]^{-1}(l'\hat{\beta} - b)}{\text{MSE}}$ Reject $H_0$ if $F > F_\alpha(1, n - p)$
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$H_0 : l'\beta = b$ vs $H_a : l'\beta \neq b$ Test Statistic: $t = \frac{l'\hat{\beta} - b}{S_{l'\hat{\beta}}}$ Reject $H_0$ if $t < -t_{\alpha/2}(n - p)$ or $t > t_{\alpha/2}(n - p)$
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are two equivalent  $\alpha$ -level tests.

**Proof.** The two test are equivalent since  $F = t^2$  and  $F_\alpha(1, n - p) = [t_{\alpha/2}(n - p)]^2$ . So we only need to show that one of the tests is  $\alpha$ -level test. We show the second one.

$$\begin{aligned} P(\text{rejecting } H_0 | H_0 \text{ is true}) &= P(t < -t_{\alpha/2}(n - p) \text{ or } t > t_{\alpha/2}(n - p) | H_0 \text{ is true}) \\ &= P(t(n - p) < -t_{\alpha/2}(n - p) \text{ or } t(n - p) > t_{\alpha/2}(n - p)) \\ &= \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha. \end{aligned}$$

(2) Tests by  $p$ -values

$H_0 : l'\beta = b \text{ vs } H_a : l'\beta \neq b$ Test Statistic: $F = \frac{(l'\hat{\beta} - b)'[l'(X'\Sigma^{-1}X)^{-1}l]^{-1}(l'\hat{\beta} - b)}{MSE}$ $p$ -value: $P(F(1, n - p) > F_{ob})$
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$H_0 : l'\beta = b \text{ vs } H_a : l'\beta \neq b$ Test Statistic: $t = \frac{l'\hat{\beta} - b}{S_{l'\hat{\beta}}}$ $p$ -value: $2P(t(n - p) >  t_{ob} )$ .
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**Ex2:** Show that  $\alpha$ -level  $F$ -test rejects  $H_0$  if and only if  $p$ -value  $< \alpha$ .

$$\begin{aligned} &\alpha\text{-level F-test rejects } H_0 \iff F_{ob} > F_\alpha(1, n - p) \\ \iff &P(F(1, n - p) > F_{ob}) < P(F(1, n - p) > F_\alpha(1, n - p)) . \\ \iff &p\text{-value} < \alpha. \end{aligned}$$

3. One-sided alternative  $t$ -tests

(1) Lower sided alternative  $t$ -test

$H_0 : l'\beta \geq b \text{ vs } H_a : l'\beta < b$ Test Statistic: $t = \frac{l'\hat{\beta} - b}{S_{l'\hat{\beta}}}$ Reject $H_0$ if $t < -t_\alpha(n - p)$	$H_0 : l'\beta \geq b \text{ vs } H_a : l'\beta < b$ Test Statistic: $t = \frac{l'\hat{\beta} - b}{S_{l'\hat{\beta}}}$ $p$ -value: $P(t(n - p) < t_{ob})$
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Show that (i) the first scheme has level  $\alpha$ .

(ii) With  $p$ -value defined in the second scheme,  $\alpha$ -level rejects  $H_0$  iff  $p$ -value  $< \alpha$ .

**Proof. (i)**

$$\begin{aligned} P(\text{rejecting } H_0 | H_0 \text{ is true}) &= P(t < -t_\alpha(n - p) | H_0 \text{ is true}) \\ &= P\left(\frac{l'\hat{\beta} - b}{S_{l'\hat{\beta}}} < -t_\alpha(n - p) | H_0 \text{ is true}\right) \\ &= P\left(\frac{l'\hat{\beta} - l'\beta}{S_{l'\hat{\beta}}} + \frac{l'\beta - b}{S_{l'\hat{\beta}}} < -t_\alpha(n - p) | H_0 \text{ is true}\right) \\ &= P\left(t(n - p) < -t_\alpha(n - p) - \frac{l'\beta - b}{S_{l'\hat{\beta}}} | l'\beta \geq b\right) \\ &\leq P(t(n - p) < -t_\alpha(n - p)) = \alpha. \end{aligned}$$

**(ii)**

$$\begin{aligned} \text{Rejecting } H_0 \text{ at the level } \alpha &\iff t_{ob} < -t_\alpha(n - p) \\ \iff P(t(n - p) < t_{ob}) &< P(t(n - p) < -t_\alpha(n - p)) \iff p\text{-value} < \alpha \end{aligned}$$

(2) Upper-sided alternative  $t$ -tests

$H_0 : l'\beta \leq b \text{ vs } H_a : l'\beta > b$ Test Statistic: $t = \frac{l'\hat{\beta} - b}{S_{l'\hat{\beta}}}$ Reject $H_0$ if $t > t_\alpha(n - p)$	$H_0 : l'\beta \leq b \text{ vs } H_a : l'\beta > b$ Test Statistic: $t = \frac{l'\hat{\beta} - b}{S_{l'\hat{\beta}}}$ $p$ -value: $P(t(n - p) > t_{ob})$
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## L15: Inference on $l'\beta$ in ANOVA and regression

### 1. $t$ -intervals in one-way ANOVA

#### (1) $t$ -intervals in one-way ANOVA

For one-way ANOVA  $Y = J\mu + \epsilon \sim N(J\mu, \sigma^2 I_n)$ ,  $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$  and  $J = \begin{pmatrix} 1_{n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1_{n_p} \end{pmatrix}$ ,  $\mu_i$ , the mean response to the treatment  $i$  corresponding to the  $i$ th level of a factor, can be written as  $l'\mu$  and thus has a  $t$ -confidence interval.

$$\mu_i \in \hat{\mu}_i \pm t_{\alpha/2}(n-p) S_{\hat{\mu}_i} = \bar{Y}_i \pm t_{\alpha/2}(n-p) \sqrt{\frac{MSE}{n_i}} \text{ where } MSE = S_{\text{pooled}}^2 = \sum_i \frac{n_i - 1}{n - p} S_i^2.$$

#### (2) SAS for CIs for $\mu_i$ .

Suppose response  $x$  and treatment `tment` (a character variable) are stored in "ex.txt". The output of the SAS

```
data a;
  infile "D:\ex.txt";
  input x tment $ @@;
proc anova;
  class tment;
  model x=tment;
  means tment/lsd alpha=0.10 clm;
run;
```

displays  $\bar{X}_i$ ,  $t_{0.05}(n-p)$ , and confidence intervals for  $\mu_i$  with confidence coefficient 0.90 for all  $i$ .

### 2. Multiple comparison in one-way ANOVA

#### (1) Confidence interval for $\mu_i - \mu_j$

$\mu_i - \mu_j$  is also of the type of  $l'\mu$ . One can also construct CI for  $\mu_i - \mu_j$

$$\mu_i - \mu_j \in \bar{Y}_i - \bar{Y}_j \pm t_{\alpha/2}(n-p) S_{\bar{Y}_i - \bar{Y}_j} \text{ where } S_{\bar{Y}_i - \bar{Y}_j}^2 = \left( \frac{1}{n_i} + \frac{1}{n_j} \right) MSE.$$

#### (2) Interpretation

$$\bar{Y}_i - \bar{Y}_j - t_{\alpha/2}(n-p) S_{\bar{Y}_i - \bar{Y}_j} > 0 \implies \mu_i > \mu_j$$

$$\bar{Y}_i + \bar{Y}_j + t_{\alpha/2}(n-p) S_{\bar{Y}_i - \bar{Y}_j} < 0 \implies \mu_i < \mu_j$$

If  $\bar{Y}_i - \bar{Y}_j - t_{\alpha/2}(n-p) S_{\bar{Y}_i - \bar{Y}_j} \leq 0$  and  $\bar{Y}_i + \bar{Y}_j + t_{\alpha/2}(n-p) S_{\bar{Y}_i - \bar{Y}_j} \geq 0$ , then no-significant difference between  $\mu_i$  and  $\mu_j$  was detected. Consequently, we conclude with  $1 - \alpha$  confidence coefficient that treatment  $i$  and treatment  $j$  have the same effects.

#### (3) Simultaneous CIs

Suppose  $\bar{Y}_1 \geq \cdots \geq \bar{Y}_p$ . By constructing  $\frac{p(p-1)}{2}$  simultaneous CIs for all  $\mu_i - \mu_j$  with  $i < j$  one can find equivalent groups for all treatments. Because all CIs were utilized to reach the conclusion, the confidence coefficient for the grouping is the overall confidence coefficient for the simultaneous CIs.

**Ex1:** Suppose  $p = 3$ . CI for  $\mu_1 - \mu_2$  did not detect significant difference; CI for  $\mu_1 - \mu_3$  shows significant difference; and CI for  $\mu_2 - \mu_3$  did not detect significant difference. The confidence coefficient for each CI is 95%. Then we can present the result by

$\mu_1 \quad \mu_2 \quad \mu_3$   
 $\text{-----}$   
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 with confidence coefficient  $1 - 3 \times 0.05 = 0.85$ .

(4) SAS

```
proc anova;
  class tment;
  model x=tment;
  means tment/bon alpha=0.10 cldiff;
run;
```

The output will display all simultaneous CIs for  $\mu_i - \mu_j$  with overall confidence coefficient 0.90. With

```
means tment/bon alpha=0.10 lines;
```

the output displays equivalent treatment groups similar to Ex1.

### 3. *t*-tests in regression

(1) For regression  $y \sim N(\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}, \sigma^2)$  with data one can do two-sided and one-sided tests on  $\beta_i$ . For example

$H_0 : \beta_i = b$ vs $H_a : \beta_i \neq b$ Test statistic: $t = \frac{\hat{\beta}_i - b}{S_{\hat{\beta}_i}}$ <i>p</i> -value: $2P(t(n-p) >  t_{ob} )$	$H_0 : \beta_i \geq b$ vs $H_a : \beta_i < b$ Test statistic: $t = \frac{\hat{\beta}_i - b}{S_{\hat{\beta}_i}}$ <i>p</i> -value: $P(t(n-p) < t_{ob})$
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These tests may have practical meanings. For example,

$H_0 : \beta_1 = 0 \iff H_0 : x_1$  is not important to the model at all

$H_0 : \beta_0 = b \implies H_0 : E(y) = b$  when  $x_1 = \dots = x_{p-1} = 0$

$H_0 : \beta_1 \leq 0 \iff H_0 : E(y)$  is a non-increasing function of  $x_1$  (Other  $x$ s are fixed)

(2) Parameter table in SAS output

`proc reg; model y=x1 x2; run;` produces parameter table

Parameter	Estimate	Standard Error	t	p
Intercept	$\hat{\beta}_0$	$s_{\hat{\beta}_0}$	$\hat{\beta}_0 / s_{\hat{\beta}_0}$	$2P(t(n-p) >  t_{ob} )$
X1	$\hat{\beta}_1$	$s_{\hat{\beta}_1}$	$\hat{\beta}_1 / s_{\hat{\beta}_1}$	$2P(t(n-p) >  t_{ob} )$
X2	$\hat{\beta}_2$	$s_{\hat{\beta}_2}$	$\hat{\beta}_2 / s_{\hat{\beta}_2}$	$2P(t(n-p) >  t_{ob} )$

(3) Making use of the parameter table

The output gives everything needed ( $t_{ob}$ , *p*-value) for testing  $H_0 : \beta_i = 0$  vs  $H_a : \beta_i \neq 0$ .

For testing on  $H_0 : \beta_i \leq 0$  vs  $H_a : \beta_i > 0$ , parameter table gives  $t_{ob}$ .

$$\text{But } p\text{-value} = \begin{cases} \frac{1}{2} [2P(t(n-p) > |t_{ob}|)] & t_{ob} > 0 \\ 1 - \frac{1}{2} [2P(t(n-p) > |t_{ob}|)] & t_{ob} < 0 \end{cases}$$

For testing on  $H_0 : \beta_i \geq 0$  vs  $H_a : \beta_i < 0$ , parameter table gives  $t_{ob}$ .

$$\text{But } p\text{-value} = \begin{cases} \frac{1}{2} [2P(t(n-p) > |t_{ob}|)] & t_{ob} < 0 \\ 1 - \frac{1}{2} [2P(t(n-p) > |t_{ob}|)] & t_{ob} > 0 \end{cases}$$

For testing on  $H_a : \beta_i \neq b$ , or  $H_a : \beta_i < b$ , or  $H_a : \beta_i > b$  parameter table gives  $\hat{\beta}_i$  and  $S_{\hat{\beta}_i}$  for you to calculate  $t_{ob}$ .