L13 Simultaneous Confidence Regions

1. Confidence regions in linear model
   Consider the model $Y = X\beta + \epsilon$, $\epsilon \sim N(0, \sigma^2\Sigma)$ where $X \in R^{n\times p}$ has full column rank.

   (1) Confidence interval for σ^2

   $$\left(\frac{\text{SSE}}{\chi^2_{\alpha/2}(n-p)}, \frac{\text{SSE}}{\chi^2_{1-\alpha/2}(n-p)}\right) \text{ is a } 1-\alpha \text{ confidence interval for } \sigma^2.$$

   (2) Confidence region for $\theta = H\beta \in R^q$.
   The collection of all $\theta \in R^q$ satisfying

   $$\frac{(\theta - H\widehat{\beta})'[H(X'\Sigma^{-1}X)^{-1}H']^{-1}(\theta - H\widehat{\beta})}{q \text{ MSE}} \le F_\alpha(q, n-p)$$

   is a $1-\alpha$ C.R. for $\theta = H\beta$ where $H \in R^{q\times p}$ has full row rank. For example

   $$\frac{(\beta - \widehat{\beta})'(X'\Sigma^{-1}X)(\beta - \widehat{\beta})}{p \text{ MSE}} \le F_\alpha(p, n-p)$$

   is $1-\alpha$ C.R. for $\beta \in R^p$.

   (3) Confidence interval for $\theta = l'\beta \in R$
   $\theta \in l'\widehat{\beta} \pm \sqrt{F_\alpha(1, n-p)}\ \sqrt{\text{MSE } l'(X'\Sigma^{-1}X)l}$ is a $1-\alpha$ C.I. for $l'\beta \in R$.
   In HW the above formula has been re-written as $\theta \in l'\widehat{\beta} \pm t_{\alpha/2}(n-p)S_{l'\widehat{\beta}}$.

2. Bonferroni simultaneous confidence regions

   (1) An inequality
   Let $A_i = [\theta_i \in B_i]$ be a random event, $i = 1, ..., k$, such that $P(A_i) \ge 1 - \frac{\alpha}{k}$.
   Then $P(A_1 \cap \cdots \cap A_i) \ge 1 - \alpha$.
   Proof.
   $$\begin{aligned} P(A_1 \cap \cdots \cap A_k) &= 1 - P(A_1^c \cup \cdots \cup A_k^c) \ge 1 - [P(A_1^c) + \cdots + P(A_k^c)] \\ &= 1 - [1 - P(A_1) + \cdots + 1 - P(A_k)] \\ &= 1 - k + [P(A_1) + \cdots + P(A_k)] \ge 1 - k + \left(1 - \tfrac{\alpha}{k}\right) k \\ &= 1 - k + k - \alpha = 1 - \alpha. \end{aligned}$$

   (2) Bonferroni method
   If $\theta_i \in B_i$ is a CR for θ_i with confidence coefficient $1 - \frac{\alpha}{k}$, $i = 1, ..., k$, then $\theta_i \in B_i$, $i = 1, ..., k$, are simultaneous CRs for θ_i, $i = 1, ..., k$, with overall confidence coefficient $1 - \alpha$.

   Ex1: Find formulas for confidence intervals for $\beta_1, ..., \beta_p$ with overall confidence coefficient 80%.
   $\beta_i \in \widehat{\beta}_i \pm t_{0.20/(2p)}(n-p)\, S_{\widehat{\beta}_i}$, $i = 1, ..., p$ are simultaneous CIs for β_i, $i = 1, ..., p$, with overall confidence coefficient 80%.

   Comment: Most stat softwares for regression produce a parameter table

Para.	Esti.	S.E.	t-value	p-value
β_0	$\widehat{\beta}_0$	$S_{\widehat{\beta}_0}$	$\widehat{\beta}_0/S_{\widehat{\beta}_0}$	$2P(t(n-p) > t - value)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
β_{p-1}	$\widehat{\beta}_{p-1}$	$S_{\widehat{\beta}_{p-1}}$	$\widehat{\beta}_{p-1}/S_{\widehat{\beta}_{p-1}}$	$2P(t(n-p) > t - value)$

3. Scheffe's simultaneous CIs

(1) Extended Cauchy-Schwartz inequality

$$0 \le (x'y)^2 \le (x'Ax)(y'A^{-1}y) \text{ where } A > 0.$$

Pf: With $A > 0$, in Cauchy-Schwartz inequality $0 \le (x'y)^2 \le (x'x)(y'y)$, replacing x by $A^{1/2}x$, and y by $A^{-1/2}y$ leads to the extended Cauchy-Schwartz inequality.

(2) A lemma: With $A > 0$,
$0 \le x'Ax \le c \Longrightarrow y'x \in \pm\sqrt{c\,(y'A^{-1}y)}$ for all $0 \ne y \in R^p$.

Pf: With $A > 0$ by the extended Cauchy-Schwartz inequality

$$\begin{aligned} 0 \le x'Ax \le c \quad &\Longrightarrow \quad 0 \le \tfrac{(y'x)^2}{y'A^{-1}y} \le x'Ax \le c \text{ for all } 0 \ne y \in R^p \\ &\Longrightarrow \quad -\sqrt{c} \le \tfrac{y'x}{\sqrt{y'A^{-1}y}} \le \sqrt{c} \text{ for all } 0 \ne y \in R^p \\ &\Longrightarrow \quad y'x \in \pm\sqrt{c\,(y'A^{-1}y)} \text{ for all } 0 \ne y \in R^p. \end{aligned}$$

(3) Scheffe's simultaneous CIs
For $Y \sim N(X\beta,\, \sigma^2\Sigma)$ where X has full column rank p, let $0 \ne l_i \in R^p$, $i = 1, 2, ...$ Then

$$l_i'\widehat{\beta} \pm \sqrt{p\,F_\alpha(p,\, n-p)}\, S_{l_i'\widehat{\beta}},\ \ i = 1, 2, ..$$

are simultaneous CIs for $l_i'\beta$, $i = 1, 2, ...$, with overall CC $1 - \alpha$.

Pf: With $\widehat{\beta} = (\Sigma^{-1/2}X)^{+}\Sigma^{-1/2}Y = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y \sim N\left(\beta,\, \sigma^2(X'\Sigma^{-1}X)^{-1}\right)$, $l_i'\widehat{\beta} \sim N\left(l_i'\beta,\, \sigma^2 l_i'(X'\Sigma^{-1}X)^{-1}l_i\right)$.
Thus $S^2_{l_i'\widehat{\beta}} = \text{MSE}\ l'(X'\Sigma^{-1}X)^{-1}l_i = l_i'A^{-1}l_i$ where $A = \frac{X'\Sigma^{-1}X}{MSE} > 0$.

Let $D = \left(\frac{(\beta-\widehat{\beta})'(X'\Sigma^{-1}X)(\beta-\widehat{\beta})}{MSE} \le p\,F_\alpha(p,\, n-p)\right) = \left((\beta - \widehat{\beta})'A(\beta - \widehat{\beta}) \le c\right)$ where $c = p\,F_\alpha(p,\, n-p)$. Then $P(D) = 1 - \alpha$.
Let $D_i = \left(l_i'\beta \in l_i'\widehat{\beta} \pm \sqrt{p\,F_\alpha(p,\, n-p)}\ S_{l_i'\widehat{\beta}}\right) = \left(l_i'(\beta - \widehat{\beta}) \in \pm\sqrt{c,\, l_i'A^{-1}l_i}\right)$.
By the extended C-S inequality,

$$(\beta - \widehat{\beta})'A(\beta - \widehat{\beta}) \le c \Longrightarrow l_i'(\beta - \widehat{\beta}) \in \sqrt{c,\, l_i'A^{-1}l_i} \text{ for all } i.$$

Thus $D \subset D_i$ for all i. So $D \subset D_1 \cap D_2 \cdots$. Hence $1 - \alpha = P(D) \le P(D_1 \cap D_2 \cdots)$.

Ex2: In $Y = X\beta + \epsilon$, $\epsilon \sim N(0, \sigma^2\Sigma)$, $Y \in R^8$, $\beta \in R^4$ and $S_{\widehat{\beta}_1} = 0.9022$.

(i) Find the width of a 95% CI for β_1.
With $\alpha = 0.05$, $\widehat{\beta}_1 \pm t_{\alpha/2}(n-p)S_{\widehat{\beta}_1}$ has width
$W = 2t_{\alpha/2}(n-p)S_{\widehat{\beta}_1} = 2\,t_{0.025}(4)\,S_{\widehat{\beta}_1} = 2 \times 2.776 \times 0.9022 = 5.009.$

(ii) Among the CIs for β_1, β_2 and β_3 with overall CC 0.95 constructed by Bonferroni method, find the width of the CI for β_1.
With $\alpha = 0.05$ and $k = 3$, $\widehat{\beta}_1 \pm t_{\alpha/(2k)}(n-p)S_{\widehat{\beta}_1}$ has width
$W = 2\,t_{\alpha/(2k)}(n-p)\,S_{\widehat{\beta}_1} = 2\,t_{0.0083}(4)\,S_{\widehat{\beta}_1} = 2 \times 3.961 \times 0.9022 = 7.1472.$

(iii) Among the CIs for all $l_i'\beta$ with overall CC 0.95 constructed by Scheffe's method, find the width of the CI for β_1.
$\widehat{\beta}_1 \pm \sqrt{pF_\alpha(p, n-p)}\,S_{\widehat{\beta}_1}$ has width
$W = 2\sqrt{pF_\alpha(p, n-p)}\,S_{\widehat{\beta}_1} = 2\sqrt{4F_{0.05}(4,\, 4)}\,S_{\widehat{\beta}_1} = 2 \times \sqrt{4 \times 6.39} \times 0.9022 = 9.1225$