L06 Best restricted estimator

1. Restricted Model

   (1) Model under restrictions
   Consider linear model (A)
   $$Y = X\beta + \epsilon \text{ with } \epsilon \sim (0,\, \sigma^2\Sigma) \text{ under the constraint } G\beta = b.$$

   (2) The restriction
   If $G\beta = b$ is consistent, then
   $$\beta \in G^+b + \mathcal{N}(G) = \{G^+b + (I - G^+G)\eta :\, \eta \in R^p\}.$$

   Proof. Suppose $G\beta = b$ has a solution β_0. Then
   $$\begin{aligned} G\beta = b &\iff G\beta = b = G\beta_0 = GG^+G\beta_0 = GG^+b \iff G(\beta - G^+b) = 0 \\ &\iff \beta - G^+b \in \mathcal{N}(G) \iff \beta \in G^+b + \mathcal{N}(G). \end{aligned}$$
   But $\mathcal{N}(G) = \mathcal{N}(G^+G) = \mathcal{R}(I - G^+G) = \{(I - G^+G)\eta :\, \eta\}$. Thus
   $$G\beta = b \iff \beta \in G^+b + \mathcal{N}(G) = \{G^+b + (I - G^+G)\eta :\, \eta\}$$

   (3) Transformed model
   Let $Y_* = Y - XG^+b$, $X_* = X(I - G^+G)$ and $\beta = G^+b + (I - G^+G)\eta$. Then Model (A) with the restriction is transformed to Model (B) $Y_* = X_*\eta + \epsilon$, $\epsilon \sim (0,\, \sigma^2\Sigma)$ without restrictions.
   Proof. (A): $\begin{cases} Y = X\beta + \epsilon,\ \epsilon \sim (0,\, \sigma^2\Sigma) \\ G\beta = b \end{cases} \iff \begin{cases} Y = X\beta + \epsilon,\ \epsilon \sim (0,\, \sigma^2\Sigma) \\ \beta = G^+b + (I - G^+G)\eta,\ \eta \in R^p \end{cases}$
   $$\iff Y = XG^+b + X(I - G^+G)\eta + \epsilon,\ \epsilon \sim (0,\ \sigma^2\Sigma)$$
   $$\iff \text{(B): } Y_* = X_*\eta + \epsilon,\ \epsilon \sim (0,\ \sigma^2\Sigma)$$

2. Restricted GLSE

   (1) Definition
   $\widehat{\beta}$ is a restricted GLSE (RGLSE) under $G\beta = b$ with respect to $U = \Sigma^{-1}$ if
   $$G\widehat{\beta} = b \text{ and } \|Y - X\widehat{\beta}\|^2_{\Sigma^{-1}} \le \|Y - X\beta\|^2_{\Sigma^{-1}} \text{ for all } \beta \text{ under } G\beta = b.$$

   (2) Formula
   Let $\widehat{\beta} = G^+b + \left[\Sigma^{-1/2}X(I - G^+G)\right]^+ \Sigma^{-1/2}(Y - XG^+b)$. Then the collection of all RGLSE under $G\beta = b$ is
   $$\text{RGLSE}_{\Sigma^{-1}}(\beta) = \widehat{\beta} + (I - G^+G)\mathcal{N}(X(I - G^+G))$$

   Proof. $\widetilde{\beta} \in \text{RGLSE}_{\Sigma^{-1}}(\beta) \overset{def}{\iff} \begin{cases} G\widetilde{\beta} = b \\ \|Y - X\widetilde{\beta}\|^2_{\Sigma^{-1}} \le \|Y - XB\|^2_{\Sigma^{-1}} \text{ for all } G\beta = b \end{cases}$
   $$\iff \begin{cases} \widetilde{\beta} = G^+b + (I - G^+G)\widetilde{\eta} \\ \|Y - X\widetilde{\beta}\|^2_{\Sigma^{-1}} \le \|Y - X\beta\|^2_{\Sigma^{-1}}\ \beta = G^+b + (I - G^+G)\eta \text{ for all } \eta \end{cases}$$
   $$\iff \begin{cases} \widetilde{\beta} = G^+b + (I - G^+G)\widetilde{\eta} \\ \|Y_* - X_*\widetilde{\eta}\|^2_{\Sigma^{-1}} \le \|Y_* - X_*\eta\|^2_{\Sigma^{-1}} \text{ for all } \eta \end{cases} \iff \begin{cases} \widetilde{\beta} = G^+b + (I - G^+G)\widetilde{\eta} \\ \widetilde{\eta} \in \text{GLSE}_{\Sigma^{-1}}(\eta) \text{ in B} \end{cases}$$
   $$\iff \begin{cases} \widetilde{\beta} = G^+b + (I - G^+G)\widetilde{\eta} \\ \widetilde{\eta} \in \left(\Sigma^{-1/2}X_*\right)^+ \Sigma^{-1/2}Y_* + \mathcal{N}(X_*) \end{cases} \iff \widetilde{\beta} \in \widehat{\beta} + (I - G^+G)\mathcal{N}\left(X(I - G^+G)\right)$$

3. Best restricted estimator

(1) Definitions

$LY + d$ is a P1UE for $H\beta$ under $G\beta = b$

$$\overset{def}{\iff} E(LY+d) = H\beta \text{ for all } \beta \text{ under consistent } G\beta = b$$

$H\beta$ is P1EU estimable under $G\beta = b$

$$\overset{def}{\iff} H\beta \text{ has a P1UE } LY+d \text{ under } G\beta = b$$

$L_0Y + d_0$ is the best P1UE for $H\beta$ under $G\beta = b$

$$\overset{def}{\iff} \begin{cases} L_0Y+d_0 \text{ is a P1UE for } H\beta \text{ under } G\beta = b \\ \mathrm{Cov}(L_0Y+d_0) \le \mathrm{Cov}(LY+d) \text{ for all P1UE } LY+d. \end{cases}$$

(2) Sufficient and necessary conditions

$$\begin{aligned} & LY+d \text{ is a P1UE for } H\beta \text{ under } G\beta = b \\ \overset{def}{\iff}\ & E(LY+d) = H\beta \text{ for all } \beta \text{ under consistent } G\beta = b \\ \iff\ & LX(I-G^+G) = H(I-G^+G) \text{ and } d = (H-LX)G^+b \end{aligned}$$

$$\begin{aligned} & H\beta \text{ is P1EU estimable under } G\beta = b \\ \iff\ & LX(I-G^+G) = H(I-G^+G) \text{ and } d = (H-LX)G^+b \text{ for some } L \text{ and } d \end{aligned}$$

(3) Best P1UE

With PIUE estimable $H\beta$ under $G\beta = b$, and $\widehat{\beta}$ in (2) of 2, $H\widehat{\beta}$ is the best restricted P1UE for $H\beta$.

Proof. $H\beta$ is P1UE estimable under $G\beta = b$. So

$H(I-G^+G) = L_1X(I-G^+G)$ and $d_1 = (H-L_1X)G^+b$ for some L_1 and d_1.

$H\widehat{\beta} = L_0Y + d_0$ where $L_0 = H\left[\Sigma^{-1/2}X(I-G^+G)\right]^+\Sigma^{-1/2}$ and $d_0 = (H-L_0X)G^+b$. But with $T = \Sigma^{-1/2}X(I-G^+G)$,

$$\begin{aligned} L_0X(I-G^+G) &= HT^+T = H(I-G^+G)T^+T = L_1X(I-G^+G)T^+T \\ &= L_1\Sigma^{1/2}TT^+T = L_1\Sigma^{1/2}T = L_1\Sigma^{1/2}\Sigma^{-1/2}X(I-G^+G) \\ &= L_1X(I-G^+G) = H(I-G^+G). \end{aligned}$$

So $H\widehat{\beta}$ is a PIUE for $H\beta$ under $G\beta = b$.

If $LY+d$ is also PIUE for $H\beta$ under $G\beta = b$, i.e., $LX(I-G^+G) = H(I-G^+G)$ and $d = (H-LX)G^+b$, we need to show

$$\mathrm{Cov}(LY+d) - \mathrm{Cov}(L_0Y+d_0) = \sigma^2(L\Sigma L' - L_0\Sigma L_0') \ge 0.$$

$$\begin{aligned} \text{Write } L_0 &= HT^+\Sigma^{-1/2} = H(I-G^+G)T^+\Sigma^{-1/2} = LX(I-G^+G)T^+\Sigma^{-1/2} \\ &= L\Sigma^{1/2}TT^+\Sigma^{-1/2} \end{aligned}$$

So $L_0\Sigma L_0' = (L\Sigma^{1/2})TT^+(L\Sigma^{1/2})'$. Therefore

$$\begin{aligned} \text{Then}\quad & \mathrm{Cov}(LY+d) - \mathrm{Cov}(L_0Y+d_0) = \sigma^2(L\Sigma L' - L_0\Sigma L_0') \\ & = \sigma^2\left(L\Sigma^{1/2}\right)(I-TT^+)\left(L\Sigma^{1/2}\right) \ge 0 \end{aligned}$$

since symmetric idempotent $I - TT^+ \ge 0$.