Stat873 HW06

For $Y = X\beta + \epsilon$, $\epsilon \sim N(0, \sigma^2\Sigma)$, $X \in R^{n\times p}$ has full column rank. $\theta = l'\beta \in R$ has BLUE $l'\widehat{\beta}$ where $\widehat{\beta}$ is the minimum norm $\text{GLSE}_{\Sigma^{-1}}(\beta)$.

1. Find $\sigma^2_{l'\widehat{\beta}}$, the variance of $l'\widehat{\beta}$.

   $\widehat{\beta} = \left(\Sigma^{-1/2}X\right)^{+}\Sigma^{-1/2}Y = \left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}Y \sim N\left(\beta, \sigma^2(X'\Sigma^{-1}X)^{-1}\right)$.
   $l'\widehat{\beta} \sim N\left(l'\beta,\ \sigma^2 l'(X'\Sigma^{-1}X)^{-1}l\right)$.
   So the variance of $l'\widehat{\beta}$ is $\sigma^2_{l'\widehat{\beta}} = \sigma^2\ l'(X'\Sigma^{-1}X)^{-1}l$.

2. Replacing parameters in the expression of $\sigma^2_{l'\widehat{\beta}}$ by their UEs one can get the estimated the variance of $l'\widehat{\beta}$, $S^2_{l'\widehat{\beta}}$. Find $S^2_{l'\widehat{\beta}}$.

   $S^2_{l'\widehat{\beta}} = \text{MSE}\ l'(X'\Sigma^{-1}X)^{-1}l$.

3. It is known that $F(1, n-p) = [t(n-p)]^2$. Derive the relation of $F_\alpha(1, n-p)$ and $t_{\alpha/2}(n-p)$.

   $$\begin{aligned}\alpha &= P\left(F(1, n-p) > F_\alpha(1, n-p)\right) = P\left([t(n-p)]^2 > F_\alpha(1, n-p)\right)\\ &= 2P\left(t(n-p) > \sqrt{F_\alpha(1, n-p)}\right).\end{aligned}$$

   So $\frac{\alpha}{2} = P\left(t(n-p) > \sqrt{F_\alpha(1, n-p)}\right)$.
   Thus $\sqrt{F_\alpha(1, n-p)} = t_{\alpha/2}(n-p)$.

4. Express the $1-\alpha$ confidence interval for $\theta = l'\beta$ derived in the lecture using the cut-off point for $t(n-p)$ distribution and $S_{l'\widehat{\beta}}$.

   $1-\alpha$ CI for $\theta = l'\beta$, $\theta \in l'\widehat{\beta} \pm \sqrt{F_\alpha(1, n-p)}\ \sqrt{\text{MSE}\ l'(X'\Sigma^{-1}X)^{-1}l}$, has been derived in the class. It can be equivalently expressed as

   $$\theta \in l'\widehat{\beta} \pm t_{\alpha/2}(n-p)S_{l'\widehat{\beta}}.$$