Stat873 HW05

1. For $\beta \in R^p$ in $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2 I_n)$, with full column rank $X \in R^{n\times p}$, $\widehat{\beta}$ is the BLUE and $\widehat{\beta}(q)$ is a principal component estimator.

   (1) Is $\widehat{\beta}(q)$ a linear biased estimator? Why?

   The principal component estimator $\widehat{\beta}(q)$ is a linear biased estimator for β.

   Let $X'X = (P_I,\, P_{II}) \begin{pmatrix} \Lambda_I & 0 \\ 0 & \Lambda_{II} \end{pmatrix} (P_I,\, P_{II})'$ be the EVD for $X'X$.

   Then $\widehat{\beta}(q) = P_I \Lambda_I^{-1} P_I' X'Y$ is a linear function of Y.

   But $E(\widehat{\beta}(q)) = P_I P_I' \beta \neq \beta$.

   So $\widehat{\beta}(q)$ is a linear biased estimator for β.

   (2) Show that $\|\widehat{\beta}(q)\|^2 \leq \|\widehat{\beta}\|^2$.

   $\widehat{\beta} = I_p \widehat{\beta} = (P_I P_I' + P_{II} P_{II}')\widehat{\beta} = P_I P_I' \widehat{\beta} + P_{II} P_{II}' \widehat{\beta}$

   But $\left\langle P_I P_I \widehat{\beta},\, P_{II} P_{II}' \widehat{\beta} \right\rangle = \widehat{\beta}' P_{II} P_{II}' P_I P_I' \widehat{\beta} = 0.$

   So $\|\widehat{\beta}\|^2 = \|P_I P_I' \widehat{\beta}\|^2 + \|P_{II} P_{II}' \widehat{\beta}\|^2 \geq \|P_I P_I' \widehat{\beta}\|^2.$

   But $P_I P_I' \widehat{\beta} = \widehat{\beta}(q).$ Hence $\|\widehat{\beta}\|^2 \geq \|\widehat{\beta}(q)\|^2.$

2. In order to find principal component estimator for β in $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2 \Sigma)$, convert the model equivalently to $Y_* = X_* \beta + \epsilon_*$, $\epsilon_* \sim (0, \sigma^2 I_n)$.

   (1) Point out the relations of Y_* and Y; X_* and X; and ϵ_* and ϵ.

   $Y_* = \Sigma^{-1/2} Y$; $X_* = \Sigma^{-1/2} X$; and $\epsilon_* = \Sigma^{-1/2} \epsilon$.

   (2) Based on the second model, results in the lecture, and relations in (1), describe the principal component estimator for β in the first model.

   By the results in the lecture, for β in the second model,

   let $X_*' X_* = (P_I,\, P_{II}) \begin{pmatrix} \Lambda_I & 0 \\ 0 & \Lambda_{II} \end{pmatrix} (P_I,\, P_{II})'$ be the EVD.

   Then $P_I \Lambda_I^{-1} P_I' X_*' Y_*$ is the principal component estimator for β.

   Thus for β in the first model,

   let $X' \Sigma^{-1} X = (P_I,\, P_{II}) \begin{pmatrix} \Lambda_I & 0 \\ 0 & \Lambda_{II} \end{pmatrix} (P_I,\, P_{II})'$ be the EVD.

   Then $\widehat{\beta}(q) = P_I \Lambda_I^{-1} P_I' X' \Sigma^{-1} Y$ is the principal component estimator for β.

3. Let $\widehat{\beta}_1$ be the BLUE of β in $Y_1 = X_1\beta + \epsilon_1$, $\epsilon_1 \sim (0, \sigma^2 I_n)$; and $\widehat{\beta}_2$ be the BLUE of β in $Y_2 = X_2\beta + \epsilon_2$, $\epsilon_2 \sim (0, \sigma^2 \Sigma)$. Write the mixed BLUE of β as weighted average of $\widehat{\beta}_1$ and $\widehat{\beta}_2$ and point out the matrix weights.

Let $W_1 = X_1' X_1$, $W_2 = X_2' \Sigma^{-1} X_2$ and $W = W_1 + W_2$.
Then the mixed BLUE is $W^{-1}(W_1 \widehat{\beta}_1 + W_2 \widehat{\beta}_2)$.