**Stat873 HW02**

1. With $E(Y_f) = H\beta \in R^q$, $\text{LUP}(Y_f) = \text{LUE}(H\beta) = \left[H(U^{1/2}X)^+U^{1/2} + \mathcal{N}(I_q,\, X)\right] Y$. Here $\mathcal{A} = H(U^{1/2}X)^+U^{1/2} + \mathcal{N}(I_q,\, X)$ is an affine set in $R^{q\times n}$.

   (1) Show that $H(U^{1/2}X)^+U^{1/2}XX^+ \in \mathcal{A}$.
   Comment: Consequently $\mathcal{A} = H(U^{1/2}X)^+U^{1/2}XX^+ + \mathcal{N}(I_q,\, X)$.

   Write $H(U^{1/2}X)^+U^{1/2}XX^+ = H(U^{1/2}X)^+U^{1/2} + Z$
   where $Z = H(U^{1/2}X)^+U^{1/2}(XX^+ - I)$.
   From $I_qZX = I_q\, H(U^{1/2}X)^+U^{1/2}(X - X) = 0, \quad Z \in \mathcal{N}(I_q,\, X)$.
   Hence $H(U^{1/2}X)^+U^{1/2}XX^+ \in H(U^{1/2}X)^+U^{1/2} + \mathcal{N}(I_q,\, X) = \mathcal{A}$

   (2) Show that $H(U^{1/2}X)^+U^{1/2}XX^+ \perp \mathcal{N}(I_q,\, X)$.
   Comment: Consequently, in $\mathcal{A}$, $H(U^{1/2}X)^+U^{1/2}XX^+$ has minimum norm.

   For $Z \in \mathcal{N}(I_q,\, X)$,

   $$\begin{aligned}\left\langle H(U^{1/2}X)^+U^{/2}XX^+,\, Z\right\rangle &= \text{tr}\left(Z'H(U^{1/2}X)^+U^{1/2}XX^+\right)\\ &= \text{tr}\left(H(U^{1/2}X)^+U^{1/2}XX^+Z'\right)\end{aligned}$$

   But $XX^+Z' = (X^+)'X'Z' = (X^+)'(I_qZX)' = 0$.
   So $\left\langle H(U^{1/2}X)^+U^{1/2}XX^+,\, Z\right\rangle = 0$ for all $Z \in \mathcal{N}(I_q,\, X)$.
   Hence $H(U^{1/2}X)^+U^{1/2}XX^+ \perp \mathcal{N}(I_q,\, X)$.

2. $\text{MSE}(\widehat{u},\, v) = E\|\widehat{u}-v\|^2 = E[(\widehat{u}-v)'(\widehat{u}-v)]$ is a real-valued risk when $v$ is predicted/estimated by $\widehat{u} \in R^q$. The matrix-valued risk in the lecture is denoted as $\text{MSEM}(\widehat{u},\, v)$.

   (1) Suppose $q = 1$. Show that $\text{MSEM}(\widehat{u},\, v) = \text{MSE}(\widehat{u},\, v)$.

   $$\begin{aligned}\text{When } q = 1,\ \text{MSEM}(\widehat{u},\, v) &= E[(\widehat{u}-v)(\widehat{u}-v)'] = E[(\widehat{u}-v)(\widehat{u}-v)]\\ &= E[(\widehat{u}-v)'(\widehat{u}-v)] = \text{MSE}(\widehat{u},\, v).\end{aligned}$$

   (2) Show that $\text{MSE}(\widehat{u},\, v) = \text{tr}\,[\text{MSEM}(\widehat{u},\, v)]$.
   Hint: $E[\text{tr}(X)] = \text{tr}[E(X)]$.

   $$\begin{aligned}\text{MSE}(\widehat{u},\, v) &= E[(\widehat{u}-v)'(\widehat{u}-v)] = E\{\text{tr}[(\widehat{u}-v)'(\widehat{u}-v)]\} = E\{\text{tr}[(\widehat{u}-v)(\widehat{u}-v)']\}\\ &= \text{tr}\{E[(\widehat{u}-v)(\widehat{u}-v)']\} = \text{tr}[\text{MSEM}(\widehat{u},\, v)].\end{aligned}$$

   (3) Show that if $\widehat{u}$ dominates $\widetilde{u}$ bu $\text{MSEM}(\cdot,\cdot)$, then $\widehat{u}$ dominates $\widetilde{u}$ by $\text{MSE}(\cdot,\cdot)$.
   Hint: $A \le 0 \Longrightarrow \text{tr}(A) \le 0$, and $A \ge 0 \Longrightarrow \text{tr}(A) \ge 0$.

   $$\begin{aligned}\widehat{u} \text{ dominates } \widetilde{u} \text{ by MSEM} &\Longrightarrow \text{MSEM}(\widetilde{u},\, v) - \text{MSEM}(\widehat{u},\, v) \ge 0\\ &\Longrightarrow \text{tr}\,[\text{MSEM}(\widetilde{u},\, v) - \text{MSEM}(\widehat{u},\, v)] \ge 0\\ &\Longrightarrow \text{MSE}(\widetilde{u},\, v) - \text{MSE}(\widehat{u},\, v) \ge 0.\\ &\Longrightarrow \widehat{u} \text{ dominates } \widetilde{u} \text{ by MSE.}\end{aligned}$$