Stat873 **HW01**

1. $\mathcal{A}$ is an affine set in linear space V. Then $0 \in \mathcal{A} \Longleftrightarrow \mathcal{A}$ is a subspace.
Show $\Rightarrow$ only.
Hint: For $x, y \in \mathcal{A}$, one needs to show $\alpha x + \beta y \in \mathcal{A}$ for all scalars α and β.
Discuss three cases: (i) $\alpha + \beta \neq 0$, (ii) $\alpha + \beta = 0$ and $\alpha = 0$, (iii) $\alpha + \beta = 0$, but $\alpha \neq 0$.

For $x, y \in \mathcal{A}$ we need to show $\alpha x + \beta y \in \mathcal{A}$ for all scalars α and β.
First we claim that $\mathcal{A}$ is closed under scalar multiplications since
$x \in \mathcal{A} \Longrightarrow \alpha x = \alpha x + (1 - \alpha)0 \in \mathcal{A}$.

(i) $\alpha + \beta \neq 0$
Affine combination $z = \frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y \in \mathcal{A}$.
By scalar multiplication, $\alpha x + \beta y = (\alpha + \beta)z \in \mathcal{A}$.

(ii) $\alpha + \beta = 0$ and $\alpha = 0$
Under $\alpha + \beta = 0$ and $\alpha = 0$, $\alpha x + \beta y = 0x + 0y = 0 \in \mathcal{A}$.

(iii) $\alpha + \beta = 0$ and $\alpha \neq 0$
Under $\alpha + \beta = 0$ and $\alpha \neq 0$, $\alpha x + \beta y = \alpha x - \alpha y$.
By scalar multiplications, $2\alpha x \in \mathcal{A}$ and $-2\alpha y \in \mathcal{A}$.
So the affine combination $\frac{1}{2}(2\alpha x) + \frac{1}{2}(-2\alpha y) = \alpha x - \alpha y \in \mathcal{A}$.

So $\mathcal{A}$ is closed under linear combinations. Hence $\mathcal{A}$ is a linear space.

2. In linear space V,

$$\mathcal{A} \text{ is an affine set} \Longleftrightarrow \mathcal{A} = x_0 + S \text{where } x_0 \in V \text{ and } S \text{ is a subspace in } V.$$

Show $\Rightarrow$ only.
Hint: Take $x_0 \in \mathcal{A}$. Then $\mathcal{A} = x_0 + \mathcal{A} - x_0$. Using 1 to show $\mathcal{A} - x_0$ is a space.

Take $x_0 \in \mathcal{A}$. Then $A = x_0 + \mathcal{A} - x_0 = x_0 + S$ where $S = \mathcal{A} - x_0$.
$x_0 \in \mathcal{A} \Longrightarrow 0 = x_0 - x_0 \in \mathcal{A} - x_0 = S$.
If $x, y \in S$, then $x = x_A - x_0$ and $y = y_A - x_0$ where $x_A, y_A \in \mathcal{A}$. But

$$\alpha x + (1 - \alpha)y = [\alpha x_A + (1 - \alpha)y_A] - x_0 \in \mathcal{A} - x_0 = S$$

So S is an affine set containing 0. By 1, S is a space.

3. $\mathcal{A} = x_0 + S$ is an affine set in V where S is a subspace. If $x_1 \in \mathcal{A}$, then $\mathcal{A} = x_1 + S$.
Show $\mathcal{A} \supset x_1 + S$ only.

If $x \in x_1 + S$, then $x = x_1 + u$ where $u \in S$.
But $x_1 \in \mathcal{A} = x_0 + S$ implies $x_1 = x_0 + u_1$ where $u_1 \in S$.
So $x = x_1 + u = x_0 + (x_1 - x_0 + u) = x_0 + (u_1 + u) \in x_0 + S = \mathcal{A}$.

4. For affine set $\mathcal{A} = x_0 + S$ in 3, $x_1 = x_0 - \pi(x_0|S) \in x_0 + S = \mathcal{A}$. So $\mathcal{A} = x_1 + S$. Show that among all $x \in \mathcal{A}$, $\|x_1\|^2 \leq \|x\|^2$.

$x \in \mathcal{A} = x_1 + S \Longrightarrow x = x_1 + u$ where $u \in S$.
But $x_1 = x_0 - \pi(x_0|S)$. So $x_1 \perp u$. By Pythagorean theorem

$$\|x\|^2 = \|x_1 + u\|^2 = \|x_1\|^2 + \|u\|^2 \geq \|x_1\|^2.$$