Stat873 **Exam 1** **Feb. 7, 2024**

Name:

1. For $Y = X\beta + \epsilon$ with $\epsilon \sim (0, \sigma^2\Sigma)$ under $G\beta = b$, the collection of all restricted generalized least square estimators for β is

$$\mathrm{RGLSE}_{\Sigma^{-1}}(\beta) = \widehat{\beta} + (I - G^+G)\mathcal{N}\left(X(I - G^+G)\right)$$

where $\widehat{\beta} = G^+b + \left[\Sigma^{-1/2}X(I - G^+G)\right]^+ \Sigma^{-1/2}(Y - XG^+b)$.
Show that $\widehat{\beta} \in \mathrm{RGLSE}_{\Sigma^{-1}}(\beta)$, and has minimum norm among all estimators in $\mathrm{RGLSE}_{\Sigma^{-1}}(\beta)$. (25 points)

Note that $0 \in \mathcal{N}(X(I - G^+G))$. So

$$\widehat{\beta} = \widehat{\beta} + 0 = \widehat{\beta} + (I - G^+G)0 \in \widehat{\beta} + (I - G^+G)\mathcal{N}\left(X(I - G^+G)\right) = \mathrm{RGLSE}_{\Sigma^{-1}}(\beta).$$

Thus $\widehat{\beta} \in \mathrm{RGLSE}_{\Sigma^{-1}}(\beta)$.

For $(I - G^+G)Z$ with $Z \in \mathcal{N}(X(I - G^+G))$, let $T = \Sigma^{-1/2}X(I - G^+G)$ and $Y_* = Y - XG^+b$.

$$\begin{aligned} \text{Then} \quad & \left\langle \widehat{\beta}, (I - G^+G)Z \right\rangle = Z'(I - G^+G)\widehat{\beta} \\ = \; & Z'(I - G^+G)G^+b + Z'(I - G^+G)\left[\Sigma^{-1/2}X(I - G^+G)\right]^+ \Sigma^{-1/2}(Y - XG^+b) \\ = \; & Z'(I - G^+G)G^+b + Z'(I - G^+G)T^+\Sigma^{-1/2}Y_*. \end{aligned}$$

Here $Z'(I - G^+G)G^+b = Z'(G^+ - G^+)b = 0$, and

$$\begin{aligned} & Z'(I - G^+G)T^+\Sigma^{-1/2}Y_* = Z'(I - G^+G)T^+TT^+\Sigma^{-1/2}Y_* \\ = \; & Z'(I - G^+G)T'(T^+)'T^+\Sigma^{-1/2}Y_* = Z'(I - G^+G)X'\Sigma^{-1/2}(T^+)'T^+\Sigma^{-1/2}Y_* \\ = \; & [X(I - G^+G)Z]'\,\Sigma^{-1/2}(T^+)'T^+\Sigma^{-1/2}Y_* = 0'\Sigma^{-1/2}(T^+)'T^+\Sigma^{-1/2}Y_* = 0. \end{aligned}$$

So $\widehat{\beta} \perp (I - G^+G)Z$ for all $Z \in \mathcal{N}(X(I - G^+G))$.

For $\widetilde{\beta} \in \mathrm{RGLSE}_{\Sigma^{-1}}(\beta) = \widehat{\beta} + (I - G^+G)\mathcal{N}(X(I - G^+G))$,

$$\widetilde{\beta} = \widehat{\beta} + (I - G^+G)Z \text{ for some } Z \in \mathcal{N}\left(X(I - G^+G)\right).$$

By Pythagorean Theorem,

$$\|\widetilde{\beta}\|^2 = \|\widehat{\beta} + (I - G^+G)Z\|^2 = \|\widehat{\beta}\|^2 + \|(I - G^+G)Z\|^2 \geq \|\widehat{\beta}\|^2.$$

Thus $\widehat{\beta} \in \mathrm{RGLSE}_{\Sigma^{-1}}(\beta)$ and has minimum norm in the class of estimators.

2. If $LY + d$ is P1UE for $H\beta$ under $G\beta = b$ in Model A: $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2\Sigma)$, then LY_* is a LUE for $H(I - G^+G)\eta$ in Model B: $Y_* = X_*\eta + \epsilon$, $\epsilon \sim (0, \sigma^2\Sigma)$ and $X_* = X(I - G^+G)$. Prove this statement. (25 points)

$LY + d$ is P1UE for $H\beta$ under $G\beta = b$ in Model A. So, by Lecture L06,

$$LX(I - G^+G) = H(I - G^+G), \text{ i.e., } LX_* = H(I - G^+G).$$

In Model B, LY_* is a LUE for $H(I - G^+G)\eta$ if and only if

$$E(LY_*) \equiv H(I - G^+G)\eta \iff LX_*\eta \equiv H(I - G^+G)\eta \text{ for all } \eta \iff LX_* = H(I - G^+G).$$

Thus LY_* is LUE for $H(I - G^+G)\eta$ in Model B.

3. For Model A: $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2\Sigma)$, under $G\beta = b$, $\beta = G^+b + (I - G^+G)\eta$. So $H\beta = HG^+b + H(I - G^+G)\eta$.
Let $\widehat{\xi}$ be the BLUE for $H(I - G^+G)\eta$ in Model B: $Y_* = X_*\eta + \epsilon$, $\epsilon \sim (0, \sigma^2\Sigma)$.
With $Y_* = Y - XG^+b$ and $X_* = X(I - G^+G)$, show that $HG^+b + \widehat{\xi}$ is the best PIUE for $H\beta$ under $G\beta = b$ in Model A. (25 points)

BLUE for $H(I - G^+G)\eta$ in Model B: $Y_* = X_*\eta + \epsilon$, $\epsilon \sim (0, \sigma^2\Sigma)$, is

$$\widehat{\xi} = H(I - G^+G)\left(\Sigma^{-1/2}X_*\right)^+ \Sigma^{-1/2}Y_*.$$

Thus

$$HG^+b + \widehat{\xi} = HG^+b + H(I - G^+G)\left(\Sigma^{-1/2}X_*\right)^+ \Sigma^{-1/2}Y_*.$$

With $Y_* = Y - XG^+b$ and $X_* = X(I - G^+G)$,

$$\begin{aligned} HG^+b + \widehat{\xi} &= HG^+b + H(I - G^+G)\left(\Sigma^{-1/2}X_*\right)^+ \Sigma^{-1/2}Y_* \\ &= HG^+b + H(I - G^+G)\left[\Sigma^{-1/2}X(I - G^+G)\right]^+ \Sigma^{-1/2}(Y - XG^+b) \\ &= HG^+b + H\left[\Sigma^{-1/2}X(I - G^+G)\right]^+ \Sigma^{-1/2}(Y - XG^+b). \end{aligned}$$

By Lecture 06, this $HG^+b + \widehat{\xi}$ is the best PIUE for $H\beta$ under $G\beta = b$.

4. Present the followings (25 points)

(1) For Model $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2\Sigma)$, write down the best PIUE for $H\beta$ under the restriction $G\beta = b$.

$HG^+b + H\left[\Sigma^{-1/2}X(I - G^+G)\right]^+\Sigma^{-1/2}(Y - XG^+b)$.

(2) For Model $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2 I_n)$, by (1) write down the best PIUE for $H\beta$ under the restriction $G\beta = b$.

$HG^+b + H\left[X(I - G^+G)\right]^+(Y - XG^+b)$.

(3) For Model $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2\Sigma)$, by (1) write down the BLUE for $H\beta$ under $G\beta = 0$.

$H\left[\Sigma^{-1/2}X(I - G^+G)\right]^+\Sigma^{-1/2}Y$.

(4) For Model $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2 I_n)$, by (1) write down the BLUE for $H\beta$ under $G\beta = 0$.

$H\left[X(I - G^+G)\right]^+Y$.

(5) For Model $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2\Sigma)$, by (1) write down the BLUE for $H\beta$.

$H\left(\Sigma^{-1/2}X\right)^+\Sigma^{-1/2}Y$.

(6) For Model $Y = X\beta + \epsilon$, $\epsilon \sim (0, \sigma^2 I_n)$, by (1) write down the BLUE for $H\beta$.

HX^+Y.