L21: Orthogonal factor models

1. Orthogonal factor models

   (1) Definition and iff conditions
   Factor model $X-\mu = LF+\epsilon$ is an orthogonal model if the columns of the loading matrix L is orthogonal, i.e., $L'L \in R^{q\times q}$ is an diagonal matrix. Thus

$$\begin{aligned} X-\mu = LF+\epsilon \text{ is an orthgonal model} &\iff L'L \text{ is a diagonal matrix} \\ &\iff L'L = \text{diag}(f_1^2, ..., f_q^2). \end{aligned}$$

   (2) An orthogonal model
   If L in factor model $X-\mu = LF+\epsilon$ is estimated by $\widehat{L} = P_I\Lambda_I^{1/2}$ where $P_I \in R^{p\times q}$ and $\Lambda_I \in R^{q\times q}$ are from the EVD $S = P\Lambda P' = (P_I,\, P_{II})\begin{pmatrix}\Lambda_I & 0\\ 0 & \Lambda_{II}\end{pmatrix}(P_I,\, P_{II})' = P_I\Lambda_I P_I' + P_{II}\Lambda_{II}P_{II}'$, then $\widehat{L}'\widehat{L} = \Lambda^{1/2}P_IP_I'\Lambda_I = \Lambda_I = \text{diag}(\lambda_1, ..., \lambda_q)$. So $X-\mu = LF+\epsilon$ is treated as an orthogonal model and $\widehat{f}_i^2 = \lambda_i$, $i = 1, ..., q$.

   (3) Distribution of estimated total variances in X
   The estimated total variances in X is
$$s_1^2 + \cdots + s_p^2 = \text{tr}(S) = \lambda_1 + \cdots + \lambda_p.$$
   The total contribution from F is
$$\textstyle\sum_{i=1}^p \widehat{h}_i^2 = \sum_{j=1}^q \widehat{f}_j^2 = \text{tr}(\widehat{L}\widehat{L}') = \text{tr}(\widehat{L}'\widehat{L}) = \lambda_1 + \cdots + \lambda_q.$$
   The total contribution from ϵ is
$$\textstyle\sum_{i=1}^p \widehat{\psi}_i = \sum_{i=1}^p (s_i^2 - \widehat{h}_i^2) = (\lambda_1 + \cdots + \lambda_p) - (\lambda_1 + \cdots + \lambda_q) = \lambda_{q+1} + \cdots + \lambda_p.$$

   Ex1: Suppose $X-\mu = LF+\epsilon$ is an orthogonal model. Convert this model to that for standardized X, $Z = V^{-1/2}(X-\mu) = (V^{-1/2}L)F + (V^{-1/2}\epsilon) = L_zF + \epsilon_z$.
   Here $L_z = V^{-1/2}L$. So $L_z'L_z = L'V^{-1/2}V^{-1/2}L = L'V^{-1}L$ may not be an diagonal matrix.
   Thus $Z = L_zF + \epsilon_z$ may not be an orthogonal model.

2. Orthogonal factor model for Z

   (1) Orthogonal factor model for Z
   If $Z = V^{-1/2}(X-\mu) = L_zF + \epsilon_z$ is an orthogonal factor model, then $\text{Cov}(Z) = \rho = L_zL_z' + \Psi_z$.
   Here ρ is the correlation matrix of X, $\Psi_z = \text{diag}(\psi_{z1}, .., \psi_{zp})$ and $L_z'L_z = \text{diag}(f_{1z}^2, ..., f_{qz}^2)$.

   (2) Estimation
   Initially estimate ρ by R, the sample correlation matrix for X. $\rho_{ii} = 1 = r_{ii}$, $i = 1, ..., p$.
   By EVD $R = P\Lambda P' = (P_I,\, P_{II})\begin{pmatrix}\Lambda_I & 0\\ 0 & \Lambda_{II}\end{pmatrix}(P_I,\, P_{II})' = P_I\Lambda_I P_I' + P_{II}\Lambda_{II}P_{II}'$.
   Estimate the main part of $\rho = L_zL_z' + \Psi_z$, L_zL_z', by the main part of R, $P_I\Lambda P_I'$, i.e., L_z is estimated by $P_I\Lambda_I^{1/2}$. Consequently $\widehat{h}_{zi}^2 = \widehat{l}_{i1}^2 + \cdots + \widehat{l}_{iq}^2$, $i = 1, ..., p$, are obtained.
   Estimated $\psi_z = 1 - h_{zi}^2$ by $\widehat{\psi}_{zi} = 1 - \widehat{h}_{zi}^2$, $i = 1, ..., p$.
   $\widehat{L}_z'\widehat{L}_z = \Lambda^{1/2}P_I'P_I\Lambda_I^{1/2} = \Lambda_I$. So $\widehat{f}_{jz}^2 = \lambda_j$, $j = 1, ..., q$.

   (3) Distribution of estimated total variances in Z
   The estimated total variances in Z is
$$r_{11} + \cdots + r_{pp} = p = \text{tr}(R) = \lambda_1 + \cdots + \lambda_p = p.$$
   The total contribution from F is
$$\textstyle\sum_{i=1}^p \widehat{h}_{zi}^2 = \sum_{j=1}^q \widehat{f}_{jz}^2 = \text{tr}(\widehat{L}_z\widehat{L}_z') = \text{tr}(\widehat{L}_z'\widehat{L}_z) = \lambda_1 + \cdots + \lambda_q.$$
   The total contribution from ϵ is
$$\textstyle\sum_{i=1}^p \widehat{\psi}_{zi} = \sum_{i=1}^p (1 - \widehat{h}_{zi}^2) = (\lambda_1 + \cdots + \lambda_p) - (\lambda_1 + \cdots + \lambda_q) = \lambda_{q+1} + \cdots + \lambda_p.$$

   Ex2: Suppose $Z = V^{-1/2}(X-\mu) = L_zF + \epsilon_z$ is an orthogonal model.
   Convert this model to that for X, $(X-\mu) = V^{1/2}Z = (V^{1/2}L_z)F + (V^{1/2}\epsilon_z) = LF + \epsilon$.
   Here $L = V^{1/2}L_z$. So $L'L = L_z'V^{1/2}V^{1/2}L_z = L_z'V^{-1}L_z$ may not be an diagonal matrix.
   Thus $X-\mu = LF+\epsilon$ may not be an orthogonal model.

3. Computations

(1) Analysis on orthogonal factor model for Z

```
data a;
   infile "D:\ex.txt";
   input x1 x2 x3 x4;
proc factor nfactor=2;
   var x1 x2 x3 x4;
   run;
```

Four parts of output

(i) Eigenvalues of correlation matrix of X. λ_i, $i = 1, .., 4$. $\lambda_1 + \cdots + \lambda_4 = 4$.

(ii) Factor pattern ($\widehat{L}_z = \left(P_1\sqrt{\lambda_1},\ P_2\sqrt{\lambda_2}\right) \in R^{4\times 2}$).

(iii) Variation explained by each factors ($\widehat{f}^2_{1z} = \lambda_1$ and $\widehat{f}^2_{2z} = \lambda_2$).

(iv) Communality ($\widehat{h}^2_{zi}$, $i = 1, .., 4$. $\sum_i \widehat{h}^2_{zi} = \sum_j \widehat{f}^2_{jz} = \lambda_1 + \lambda_2$.

(2) Directly entering S

In 9.8 on p531 $S = \begin{pmatrix} 1 & 0.4 & 0.9 \\ 0.4 & 1 & 0.7 \\ 0.9 & 0.7 & 1 \end{pmatrix}$

(i) Entering S

```
data a (type='cov');
   _TYPE_='COV';
   input _NAME_ $ x1 x2 x3;
   datalines;
x1 1   0.4  0.9
x2 0.4  1   0.7
x3 0.9 0.7   1
;
```

```
data a (type='cov');
   _TYPE_='COV';
   input _NAME_ $ x1 x2 x3;
   datalines;
x1 1  0.4  0.9
x2 .   1   0.7
x3 .   .    1
;
```

```
data a (type='cov');
   _TYPE_='COV';
   input _NAME_ $ x1 x2 x3;
   datalines;
x1 1     .     .
x2 0.4  1      .
x3 0.9 0.7     1
;
```

(ii) Analysis on orthogonal model for X

```
proc factor nfactor=1 cov;
    var x1 x2 x3;
    run;
```

(iii) Analysis on orthogonal factor model for Z

```
proc factor nfactor=1;
    var x1 x2 x3;
    run;
```