L12 One-sample test

1. Test using rejection rule

   (1) Test scheme
   For hypotheses on μ in $N(\mu, \Sigma)$, based on a sample of size n there is a 3-step test scheme.

   $H_0: \mu = \mu_0$ vs $H_a: \mu \neq \mu_0$
   Test Statistic: $T^2 = (\overline{X} - \mu_0)' \left(\frac{S}{n}\right)^{-1} (\overline{X} - \mu_0)$
   Reject H_0 if $T^2 > T^2_\alpha(p, n-1)$

   (2) Significance level
   Rejecting H_0 when H_0 is true is Type I error. Accepting H_0 when H_0 is false is Type II error. The largest probability of Type I error is the size of the test.
   Statistical test is designed to control the size of the test. If the size is controlled to be $\leq \alpha$, then α is called the significance level of the test. The test is an α-level test.

   (3) Test in (1) is an α-level test.
   Proof. $$\begin{aligned} P(\text{Type I error}) &= P(\text{Rejecting } H_0 | H_0 \text{ is true}) = P(T^2 > T^2_\alpha(p, n-1) | \mu = \mu_0) \\ &= P(T^2(p, n-1) > T^2_\alpha(p, n-1)) = \alpha. \end{aligned}$$

   Ex1: To present a specific test scheme μ_0 must be specified, $T^2_\alpha(p, n-1)$ must be specified as well. For example for testing $\mu = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$ at the level 0.05 based on a sample of size $n = 3$,
   $T^2_\alpha(p, n-1) = \frac{(n-1)p}{n-p} F_\alpha(p, n-p) = \frac{2\times 2}{1} F_{0.05}(2, 1) = 4 \times 199.5 = 798$. So

   $H_0: \mu = \mu_0$ vs $H_a: \mu \neq \mu_0$ where $\mu_0 = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$
   Test Statistic: $T^2 = (\overline{X} - \mu_0)' \left(\frac{S}{3}\right)^{-1} (\overline{X} - \mu_0)$
   Reject H_0 if $T^2 > 798$ for $\alpha = 0.05$

2. Likelihood ratio test

   (1) Likelihood ratio test (LRT)
   With likelihood function $L(\mu, \Sigma)$, $\frac{\max[L(\mu_0, \Sigma):\, \Sigma]}{\max[L(\mu, \Sigma):\, \mu, \Sigma]}$ is the likelihood ratio (LR).
   By intuition H_0 should be rejected for smaller value of LR. A such test is LRT.
   If LR is an increasing (decreasing) function of statistic T, then T can be used as a test statistic and H_0 is rejected for smaller (larger) values of T.

   (2) Test in (1) of 1 is a LRT
   By algebraic manipulation,
   $\max[L(\mu_0, \Sigma) : \Sigma] = \left(\frac{n}{2\pi e}\right)^{(np)/2} |E_0|^{-n/2}$ where $E_0 = \sum_i (X_i - \mu_0)(X_i - \mu_0)'$.
   $\max[L(\mu, \Sigma) : \mu, \Sigma] = \left(\frac{n}{2\pi e}\right)^{(np)/2} |E|^{-n/2}$ where $E = \text{CSSCP}$.
   So LR$= \left(\frac{|E|}{|E_0|}\right)^{n/2}$ is an increasing function of $\Lambda = \frac{|E|}{|E_0|}$ called Wilks Lambda, and
   $\Lambda = \left(1 + \frac{T^2}{n-1}\right)^{-1}$ is a decreasing function of $T^2 = (\overline{X} - \mu_0)' \left(\frac{S}{n}\right)^{-1} (\overline{X} - \mu_0)$. Thus

   $H_0: \mu = \mu_0$ versus $H_a: \mu \neq \mu_0$
   Test Statistic $\Lambda = \frac{|E|}{|E_0|}$
   Reject H_0 if $\Lambda < c_1$

   and test in 1 (1) are both LRTs.

(3) Remarks
Two-step implementation: Present calculated value of test statistic and state the conclusion.
Two conclusions: (i) Reject H_0. The error probability is controlled by the significance level.
(ii) Fail to reject H_0. The probability of error called Type II error has not been controlled.
Relation $\Lambda = \left(1 + \frac{T^2}{n-1}\right)^{-1} \iff T^2 = \left(\frac{1}{\Lambda} - 1\right)(n-1)$ can be used to get T^2_{ob}.

Ex2: If $\Lambda = 0.72$ and $n = 3$, then $T^2 = 0.7778$. So we have the report on the test,

> $H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$ where $\mu_0 = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$
> Test Statistic: $T^2 = (\overline{X} - \mu_0)' \left(\frac{S}{3}\right)^{-1} (\overline{X} - \mu_0)$
> Reject H_0 if $T^2 > 798$ for $\alpha = 0.05$
>
> $T^2_{ob} = 0.7778$
> Fail to reject H_0.

3. Test by p-value

(1) p-value: Observed significance level
Based on observed T^2_{ob}, the smallest significance level that allows H_0 to be rejected is called the p-value or the observed significance level.
High p-value means high error probability if H_0 is rejected. Thus p-value is the degree of consistency of data with H_0.
With p-value, the universal rejection rule is to reject H_0 if p-vlue$< \alpha$.

(2) Test scheme using p-value
has three-steps: Hypotheses, test statistic, and the formula for p-value

> $H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$
> Test statistic: $T^2 = (\overline{X} - \mu_0)' \left(\frac{S}{n}\right)^{-1} (\overline{X} - \mu_0)$
> p-value: $P(T^2(p, n-1) > T^2_{ob})$

(3) Implementation
Implementation still has two steps: Computation and conclusion.
Computation includes the computation for T^2_{ob} and the computation for p-value,
$P(T^2(p, n-1) > T^2_{ob}) = P\left(\frac{(n-1)p}{n-p} F(p, n-p) > T^2_{ob}\right) = P\left(F(p, n-p) > \frac{n-p}{(n-1)p} T^2_{ob}\right)$.

Ex3: For data and hypotheses in Ex2, using p-value

> $H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$ where $\mu_0 = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$
> Test Statistic $T^2 = (\overline{X} - \mu_0)' \left(\frac{S}{n}\right)^{-1} (\overline{X} - \mu_0)$
> p-value: $P(T^2(2, 2) > T^2_{ob})$
>
> $T^2 = \left(\frac{1}{0.72} - 1\right) \times 2 = 0.7778$;
> p-value: $P(T^2(2, 2) > 0.7778) = P(F(2, 1) > 0.19) = 0.8485$
> Fail to reject H_0

L13: SAS for one-sample test

1. Data
   For one-sample test by rejection region we need to calculate T^2_{ob}. For one-sample test by p-value, besides T^2_{ob} we also need to calculate p-value: $P(T^2(p, n-1) > T^2_{ob}) = P\left(F(p, n-p) > \frac{n-p}{(n-1)p} T^2_{ob}\right)$.

   (1) Enter sample into SAS
   Suppose based on sample $X_1 = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$, $X_2 = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$, $X_3 = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$ we want to test $\mu = \mu_0 = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

```
data a;
   input x1 x2 @@;
   datalines;
   6 9 10 6 8 3
   ;
```

or

```
data a;
   infile "D://example.txt";
   input x1 x2 @@;
```

   (2) Modify data
   SAS will test $H_0 : \mu = 0$. To test $H_0 : \mu = \mu_0$, note that $\mu = \mu_0 \Longleftrightarrow \mu - \mu_0 = 0$. So we need to convert the population to the one with mean $\mu - \mu_0$. The sample from this new population has sample $X_i - \mu_0$, $i = 1, ..., n$ where X_i, $i = 1, ..., n$, is the original sample.

```
data b;
   set a;
   y1=x1-9;
   y2=x2-6;
```

or in one step

```
data a;
   infile "D://example.txt";
   input x1 x2 @@;
   y1=x1-9; y2=x2-6;
```

2. Procedure and output

   (1) proc reg and its output

```
proc reg;
    model y1 y2=/noprint;
 mtest intercept;
 run;
```

Statistics	Value	F-value	Num DF	Den DF	Pr>F
Wilks' Lambda	0.7200	0.19	2	1	0.8485
Pillai's Trace	0.2800	0.19	2	1	0.8485
Hotellig-Lawley Trace	0.3889	0.19	2	1	0.8485
Roy's Greatest Root	0.3889	0.19	2	1	0.8485

   $T^2_{ob} = \left(\frac{1}{\Lambda} - 1\right)(n-1)$ can be calculated based on Wilk's Lambda, Λ, given in the output.
   $F_{ob} = \frac{n-p}{(n-1)p} T^2_{ob}$ is given in the output, p-value: $P(F(p, n-p) > F_{ob})$ is also given int the output.

   (2) Testing on $H_0 : \mu = \mu_0$ using rejection region

   $H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$ where $\mu_0 = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$
   Test Statistic: $T^2 = (\overline{X} - \mu_0)' \left(\frac{S}{3}\right)^{-1} (\overline{X} - \mu_0)$
   Reject H_0 if $T^2 > 798$ for $\alpha = 0.05$
   $T^2_{ob} = 0.7778$
   Fail to reject H_0.

(2) Using p-value

> $H_0: \mu = \mu_0$ versus $H_a: \mu \neq \mu_0$ where $\mu_0 = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$
>
> Test Statistic $T^2 = (\overline{X} - \mu_0)' \left(\frac{S}{n}\right)^{-1} (\overline{X} - \mu_0)$
> p-value: $P(T^2(2,\,2) > T^2_{ob})$
>
> $T^2 = \left(\frac{1}{0.72} - 1\right) \times 2 = 0.7778$;
> p-value: $P(T^2(2,2) > 0.7778) = P(F(2,1) > 0.19) = 0.8485$
> Fail to reject H_0

3. Four test statistics in SAS output

(1) Wilk's Lambda
$\Lambda = \frac{|E|}{|E_0|} = \frac{|E|}{|E+H|}$ where the error matrix $E = \sum_i (X_i - \overline{X})(X_i - \overline{X})' = \text{CSSCP}$.
$E_0 = \sum_i (X_i - \mu_0)(X_i - \mu_0)' = \sum_i (X_i - \overline{X} + \overline{X} - \mu_0)(X_i - \overline{X} - \overline{X} - \mu_0)' = E + n(\overline{X} - \mu_0)(\overline{X} - \mu_0)'$
$= E + H$ where $H = n(\overline{X} - \mu_0)(\overline{X} - \mu_0)'$.
Recall: $T^2 = (\overline{X} - \mu_0)\left(\frac{S}{n}\right)^{-1}(\overline{X} - \mu_0) = n(n-1)(\overline{X} - \mu_0)' E^{-1} (\overline{X} - \mu_0)$.
By formula $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{11}| \cdot |A_{22.1}| = |A_{22}| \cdot |A_{11.2}|$ where $A_{11.2} = A_{11} - A_{12} A_{22}^{-1} A_{21}$ and $A_{22.1} = A_{22} - A_{21} A_{11}^{-1} A_{12}$, from $\begin{vmatrix} 1 & n(\overline{X} - \mu_0)' \\ -(\overline{X} - \mu_0) & E \end{vmatrix}$,
$|E + n(\overline{X} - \mu_0)(\overline{X} - \mu_0)'| = |E| \left[1 + n(\overline{X} - \mu_0)' E^{-1} (\overline{X} - \mu_0)\right]$, i.e., $|E + H| = |E| \left(1 + \frac{T^2}{n-1}\right)$.
So $\Lambda = \left(1 + \frac{T^2}{n-1}\right)^{-1}$. Thus $T^2 = \left(\frac{1}{\Lambda} - 1\right)(n-1)$.

(2) Hotelling-Lawley trace
H-L trace $\stackrel{def}{=} \text{tr}(HE^{-1}) = \text{tr}\left[n(\overline{X} - \mu_0)(\overline{X} - \mu_0)' E^{-1}\right] = n(\overline{X} - \mu_0)' E^{-1} (\overline{X} - \mu_0) = \frac{T^2}{n-1}$.
So $T^2 = (n-1)$H-L trace.

(3) Roy's greatest root
Roy's greatest root is the largest eighenvalue of $E^{-1/2} H E^{-1/2}$.
Let $E^{-1/2} H E^{-1/2} = Q \Gamma Q'$ be the EVD where $\Gamma = \text{diag}(\gamma_1, .., \gamma_p)$, $\gamma_1 \geq \cdots \geq \gamma_p$.
Then Roy's greatest root is γ_1.
But rank $\left(E^{-1/2} H E^{-1/2}\right) = \text{rank}(H) = \text{rank}\left[n(\overline{X} - \mu_0)(\overline{X} - \mu_0)'\right] = 1$.
Thus $E^{-1/2} H E^{-1/2}$ has only one non-zero eigenvalue γ_1. So

$$\begin{aligned} \text{Roy's greatest root} &= \gamma_1 = \gamma_1 + \cdots + \gamma_p = \text{tr}\left(E^{-1/2} H E^{-1/2}\right) = \text{tr}\left(HE^{-1}\right) \\ &= \text{H-L trace} = \frac{T^2}{n-1}. \end{aligned}$$

(4) Pillai's trace

$$\begin{aligned} \text{Pillai's trace} &\stackrel{def}{=} \text{tr}[H(E+H)^{-1}] = \text{tr}\left[E^{1/2}\left(E^{-1/2} H E^{-1/2}\right) E^{1/2} (E+H)^{-1}\right] \\ &= \text{tr}\left[\left(E^{-1/2} H E^{-1/2}\right) E^{1/2} (E+H)^{-1} E^{1/2}\right] \\ &= \text{tr}\left[Q\Gamma Q' \left(I + E^{-1/2} H E^{-1/2}\right)^{-1}\right] = \text{tr}\left[Q\Gamma Q' \left(I + Q\Gamma Q'\right)^{-1}\right] \\ &= \text{tr}\left\{Q\Gamma Q' \left[Q(I+\Gamma)Q'\right]^{-1}\right\} = \text{tr}\left[Q\Lambda Q' Q (I+\Gamma)^{-1} Q'\right] \\ &= \text{tr}\left[\Gamma(I+\Gamma)^{-1}\right] = \frac{\gamma_1}{1+\gamma_1} + \cdots + \frac{\gamma_p}{1+\gamma_p} = \frac{\gamma_1}{1+\gamma_1} = 1 - (1+\gamma_1)^{-1} \\ &= 1 - [1 + (\text{H-L trace})]^{-1} = 1 - \left(1 + \frac{T^2}{n-1}\right)^{-1}. \end{aligned}$$

Thus $T^2 = \left(\frac{1}{1 - \text{Pillai trace}} - 1\right)(n-1)$.

Comment: Wilk Lambda + Pillai trace= 1
Hotelling-Lawley trace= Roy's greatest root.