Stat776 **HW09**

1. The following is Factor model analysis table.

X_i	σ_i^2	F_1	F_2	h_i^2	ψ_i
X_1	19	16	1	17	2
X_2	57	49	4	53	4
X_3	38	1	36	37	1
X_4	68	1	64	65	3
	182	67	105	172	10

(1) Find the contribution of factor F to $\text{var}(X_2)$.

The contribution of factor F to $\text{var}(X_2)$ is $h_2^2 = 53$.

(2) Find the part of the total variance in X explained by F_2.

The part of the total variances in X explained by F_2 is $f_2^2 = 105$.

(3) Let Z be standardized X. Find the part of $\text{var}(Z_3) = 1$ contributed by F_2.

The part of $\text{var}(Z_3) = 1$ contributed by F_2 is
$l_{z32}^2 = l_{32}^2/\sigma_3^2 = \frac{36}{38} = \frac{18}{19} = 0.9474$.

2. In 9.8 on page 531 $\text{Cov}(X)$, $\Sigma = \begin{pmatrix} 1 & 0.4 & 0.9 \\ 0.4 & 1 & 0.7 \\ 0.9 & 0.7 & 1 \end{pmatrix}$ is given.
Show that for this X there is no factor model $X - \mu = LF + \epsilon$ with $L \in R^{3\times 1}$.

If $X - \mu = LF + \epsilon$ with $L = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$, then $\Sigma = LL' + \Psi$, i.e.,

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0.4 & 0.9 \\ 0.4 & 1 & 0.7 \\ 0.9 & 0.7 & 1 \end{pmatrix} = \begin{pmatrix} l_1^2 + \psi_1 & l_1 l_2 & l_1 l_3 \\ l_2 l_1 & l_2^2 + \psi_2 & l_2 l_3 \\ l_3 l_1 & l_3 l_2 & l_3^2 + \psi_3 \end{pmatrix}$$

So $\psi_3 = \sigma_3^2 - l_3^2 = \sigma_3^2 - \frac{\sigma_{12}\times\sigma_{13}\times\sigma_{23}}{(\sigma_{12})^2} = -\frac{23}{40} < 0$ that contradicts $\psi_3 = \text{var}(\epsilon_3) \geq 0$.
Here the factor model does not exist.

3. Table 9-12 on page 536 is stored in T9-12.dat. Run

`data a;`
`infile "D:\T9-12.dat";`
`input x1 x2 x3 x4 x5 x6 x7;`
`run;`
	`proc factor nfactor=2 cov;`
	`var x1 x2 x3;`
	`run;`

Consider factor model $X - \mu = LF + \epsilon$ with $\epsilon \sim (0, \Psi)$ and factor model for standardized X, $Z = L_z F + \epsilon_z$ with $\epsilon_z \sim (0, \Psi_z)$. For the following computation problems keep 5 digits after decimal point for final results.

(1) Find $\widehat{\Psi}_z$, the estimated $\Psi_z = \text{Cov}(\epsilon_z)$.

$$\widehat{\Psi}_z = \begin{pmatrix} 1-\widehat{h}_{z1}^2 & 0 & 0 \\ 0 & 1-\widehat{h}_{z2}^2 & 0 \\ 0 & 0 & 1-\widehat{h}_{z3}^2 \end{pmatrix} = \begin{pmatrix} 1-0.97126 & 0 & 0 \\ 0 & 1-0.99888 & 0 \\ 0 & 0 & 1-0.02488 \end{pmatrix}$$

$$= \begin{pmatrix} 0.02874 & 0 & 0 \\ 0 & 0.00112 & 0 \\ 0 & 0 & 0.07512 \end{pmatrix}$$

(2) Find $\widehat{L}$, the estimated loading matrix L.

$$\widehat{L} = \widehat{V}^{1/2}\widehat{L}_z = \begin{pmatrix} \sqrt{53.83664} & 0 & 0 \\ 0 & \sqrt{102.50175} & 0 \\ 0 & 0 & \sqrt{22.20500} \end{pmatrix} \begin{pmatrix} 0.97068 & 0.17040 \\ 0.98724 & -0.15570 \\ 0.89828 & 0.34348 \end{pmatrix}$$

$$= \begin{pmatrix} 7.12221 & 1.25028 \\ 9.99513 & -1.57636 \\ 4.23289 & 1.61855 \end{pmatrix}$$

(3) Find $\widehat{\Psi}$, the estimated $\Psi = \text{Cov}(\epsilon)$.

$$\widehat{\Psi} = \begin{pmatrix} s_1^2(1-\widehat{h}_{z1}^2) & 0 & 0 \\ 0 & s_2^2(1-\widehat{h}_{z2}^2) & 0 \\ 0 & 0 & s_3^2(1-\widehat{h}_{z3}^2) \end{pmatrix}$$

$$= \begin{pmatrix} 50.83664(1-0.97125) & 0 & 0 \\ 0 & 102.50175(1-0.99888) & 0 \\ 0 & 0 & 22.20500(1-0.92488) \end{pmatrix}$$

$$= \begin{pmatrix} 1.54727 & 0 & 0 \\ 0 & 0.11480 & 0 \\ 0 & 0 & 1.66804 \end{pmatrix}$$