Stat776 **HW04**

1. Suppose $X_1, ..., X_m$ is a random sample from $N(\mu_x, \Sigma)$ and $Y_1, ..., Y_n$ is a random sample from $N(\mu_y, \Sigma)$. Let $Z = (X_1, ..., X_m, Y_1, .., Y_n) \in R^{p\times(m+n)}$. Express the distribution of Z.

$$Z \sim N_{p\times(m+n)}((\mu_x 1_m', \mu_y 1_n'), \Sigma, I_{m+n}) = N_{p\times(m+n)}\left((\mu_x, \mu_y)\begin{pmatrix} 1_m' & 0 \\ 0 & 1_n' \end{pmatrix}, \Sigma, I_{m+n}\right).$$

2. Find the following probabilities.

(1) For $X \sim W_{1\times1}(5)$, find $P(X > 12)$.

$X \sim W_{1\times1}(5) \Longrightarrow X \sim \chi^2(5)$.
So $P(X > 12) = P(\chi^2(5) > 12) = 0.03479$

(2) For $X \sim W_{1\times1}(5, 4)$, find $P(X > 30)$.

$X \sim W_{1\times1}(5, 4) \Longrightarrow \frac{X}{4} = \frac{1}{2}X\frac{1}{2} \sim \frac{1}{2}W_{1\times1}(5, 4)\frac{1}{2} = W_{1\times1}(5, 1) = W_{1\times1}(5) = \chi^2(5)$.
So $P(X > 30) = P\left(\frac{X}{4} > \frac{30}{4}\right) = P(\chi^2(5) > 7.5) = 0.186$

(3) $P(T^2(5, 14) > 3)$.

$T^2(5, 14) = \frac{5\times14}{14-5+1}F(5, 14-5+1) = 7F(5, 10)$
So $P(T^2(5, 14) > 3) = P(7F(5, 10) > 3) = P(F(5, 10) > 0.4286) = 0.8189$

3. Let $\overline{X} \in R^4$ and $S \in R^{4\times4}$ be from a sample of size 20 from $N(\mu, \Sigma)$.
Define $Y = (\overline{X} - \mu)'\left(\frac{S}{20}\right)^{-1}(\overline{X} - \mu)$. Find $P(Y > 4)$.

$Y = (\overline{X} - \mu)'\left(\frac{S}{20}\right)^{-1}(\overline{X} - \mu) \sim T^2(4, 19) = \frac{4\times19}{19-4+1}F(4, 19-4+1) = \frac{19}{4}F(4, 16)$.
$P(Y > 4) = P(T^2(4, 19) > 4) = P\left(\frac{19}{4}F(4, 16) > 4\right) = P(F(4, 16) > 0.8421) = 0.5186$

4. Suppose $X \sim N_4(\mu, 4\Sigma)$ is independent to $W \sim W_{4\times4}(16, \Sigma)$. In the following expression find a, b and c.

$$(X - \mu)'\left(\frac{W}{a}\right)^{-1}(X - \mu) \sim T^2(b, c)$$

$X \sim N_4(\mu, 4\Sigma) \Longrightarrow \frac{X-\mu}{2} \sim N(0, \Sigma)$. So $\frac{X-\mu}{2} \sim N(0, \Sigma)$ is independent to $W \sim W_{4\times4}(16, \Sigma)$.

Thus $\left(\frac{X-\mu}{2}\right)'\left(\frac{W}{16}\right)^{-1}\left(\frac{X-\mu}{2}\right) \sim T^2(4, 16)$, i.e., $(\overline{X} - \mu)'\left(\frac{W}{4}\right)^{-1}(\overline{X} - \mu) \sim T^2(4, 16)$.

Thus $a = 4$, $b = 4$ and $c = 16$.