Stat776 HW02

1. p201 4.3 (c) (d) (e)

$$\Sigma = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) $\text{Cov}\left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, X_3\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

Thus $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and X_3 are independent.

(d) $\text{cov}\left(\frac{X_1+X_2}{2}, X_3\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.$

Thus $\frac{X_1+X_2}{2}$ and X_3 are independent.

(e) $\text{cov}\left(X_2, X_2 - \frac{5}{2}X_1 - X_3\right) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -\frac{5}{2} \\ 1 \\ -1 \end{pmatrix} = 10.$

Thus X_2 and $X_2 - \frac{5}{2}X_1 - X_3$ are not independent.

2. p201 4.4 (b)

(b) Find a such that X_2 and $X_2 - a'\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ are independent.

$0 = \text{Cov}\left[X_2, X_2 - a'\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} -a_1 \\ 1 \\ -a_3 \end{pmatrix} = -a_1 + 3 - 2a_3$

$\Rightarrow a_1 = 3 - 2a_3 \Rightarrow a = \begin{pmatrix} 3 - 2a_3 \\ a_3 \end{pmatrix}$. For example with $a = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ X_2 and $X_2 - a'\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ are independent.

3. p470 8.1

$0 = |\Sigma - \lambda I| \Rightarrow \lambda_1 = 6, \lambda_2 = 1$

$(\Sigma - \lambda_1 I)x = 0 \Leftrightarrow x_1 = 2x_2 \Leftrightarrow x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} x_2 \Rightarrow P_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

$(\Sigma - \lambda_2 I)x = 0 \Leftrightarrow x_2 = -2x_{\Leftrightarrow} x = \begin{pmatrix} 1 \\ -2 \end{pmatrix} x_1 \Rightarrow P_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

$Y_1 = \frac{2}{\sqrt{5}}X_1 + \frac{1}{\sqrt{5}}X_2$ and $Y_2 = \frac{1}{\sqrt{5}}X_1 - \frac{2}{\sqrt{5}}X_2$.

$\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{6}{7} = 85.71\%$ of total variances are explained by Y_1.

4. p470 8.2 (a) Do not calculate the principal components Y_1 and Y_2, but calculate the proportion of total variance in ρ explained by Y_1.

$\Sigma = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$, $V = \begin{pmatrix} 5 & 9 \\ 0 & 2 \end{pmatrix}$, $\rho = V^{-1/2}\Sigma V^{-1/2} = \begin{pmatrix} 1 & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & 1 \end{pmatrix}$.

$|\rho - \lambda I| = \begin{vmatrix} 1-\lambda & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - \left(\frac{2}{\sqrt{10}}\right)^2 = \left(\lambda - 1 - \frac{2}{\sqrt{10}}\right)\left(\lambda - 1 + \frac{2}{\sqrt{10}}\right)$.

Thus $\lambda_1 = 1 + \frac{2}{\sqrt{10}}$ and $\lambda_2 = 1 - \frac{2}{\sqrt{10}}$.

The proportion of total variance in ρ explained by Y_1 is $\frac{\lambda_1}{2} = \frac{1+\frac{2}{\sqrt{10}}}{2} = 0.8162$.