Stat 776 HW01

1. $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right)$.

   (1) Find the distribution of $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} = \begin{pmatrix} e_1' \\ e_3' \end{pmatrix} X \sim N\left(\begin{pmatrix} e_1' \\ e_3' \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} e_1' \\ e_3' \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}(e_1 \quad e_3)\right)$$

$$= N\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

   (2) Find the distribution of $X_1 \left| \begin{pmatrix} X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}\right.$

$$X_1 \left| \begin{pmatrix} X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}\right. \sim N\left(1 + (0,\, 1)I^{-1}\begin{pmatrix} 5-2 \\ 6-3 \end{pmatrix}, 2 - (0,\, 1)I^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$
$$= N(4,\, 1^2)$$

   (3) Find the conditional probability of $X_1 > 4$ given $\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, i.e., $P(X_1 > 4 | X_2 = 5,\, X_3 = 6)$.

   Let $Y \sim (4,\, 1^2)$ and $Z \sim N(0,\, 1^2)$.
   Then $P(X_1 > 4 | X_2 = 5,\, X_3 = 6) = P(Y > 4) = P(Z > 0) = 0.5$.

2. p107 2.30 (a)-(i)

   (a) $E(X^{(1)}) = E\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} X\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\mu_X = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

   (b) $E(AX^{(1)}) = AE(X^{(1)}) = 10$

   (c) $\text{Cov}(X^{(1)}) = \text{Cov}\left((I_2 \quad 0)\, X\right) = (I_2 \quad 0)\,\Sigma\begin{pmatrix} I_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

   (d) $\text{Cov}(AX^{(1)}) = A\text{Cov}(X^{(1)})A' = 7$

   (e) $E(X^{(2)}) = E\left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} X\right] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\mu_X = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(f) $E(BX^{(2)}) = BE(X^{(2)}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

(g) $\mathrm{Cov}(X^{(2)}) = \mathrm{Cov}\left(\begin{pmatrix} 0 & I_2 \end{pmatrix} X\right) = \begin{pmatrix} 0 & I_2 \end{pmatrix} \Sigma \begin{pmatrix} 0 \\ I_2 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix}$

(h) $\mathrm{Cov}(BX^{(2)}) = B\mathrm{Cov}(X^{(2)})B' = \begin{pmatrix} 33 & 36 \\ 36 & 48 \end{pmatrix}$

(i) $\mathrm{Cov}(X^{(1)}, X^{(2)}) = \mathrm{Cov}\left(\begin{pmatrix} I_2 & 0 \end{pmatrix} X,\ \begin{pmatrix} 0 & I_2 \end{pmatrix} X\right) = \begin{pmatrix} I_2 & 0 \end{pmatrix} \Sigma \begin{pmatrix} 0 \\ I_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$

3. p201 4.4 (a) Find the distribution of $3X_1 - 2X_2 + X_3$

$$\begin{aligned} 3X_1 - 2X_2 + X_3 &= (3,\ -2,\ 1)X \sim N\left((3,\ -2,\ 1)\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},\ (3,\ -2,\ 1)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\right) \\ &= N(13,\ 3^2) \end{aligned}$$

4. p202 4.7 (b)

$$\begin{aligned} & X_1 \left| \begin{pmatrix} X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \right. \\ \sim\ & N\left(1 + \begin{pmatrix} 0 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}^{-1}\begin{pmatrix} x_2 + 1 \\ x_3 - 2 \end{pmatrix},\ 4 - \begin{pmatrix} 0 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) \\ =\ & N\left(2 - \tfrac{x_3}{2},\ \tfrac{7}{2}\right) \end{aligned}$$