

Name : _____

Score : _____

1. $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 & -1 \\ -1 & 10 \end{pmatrix} \right)$. (30 points)

(1) Find $E(X_1|X_2)$.

$$E(X_1|X_2) = 1 + (-1)\frac{1}{10}(X_2 - 2) = \frac{12-X_2}{10}$$

(2) Find $\text{var}(X_2|X_1)$.

$$\text{var}(X_2|X_1) = 10 - (-1)\frac{1}{5}(-1) = \frac{49}{5}$$

(3) Find $\text{Cov} \left(\begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \end{pmatrix}, 2X_1 - X_2 \right)$.

$$\begin{aligned} \text{Cov} \left(\begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \end{pmatrix}, 2X_1 - X_2 \right) &= \text{Cov} \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} X, (2, -1)X \right) \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ -1 & 10 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 23 \end{pmatrix} \end{aligned}$$

(4) $X_1 + X_2$ and $cX_1 - X_2$ are independent. Find c .

$$0 = \text{Cov}((1, 1)X, (c, -1)X) = (1, 1) \begin{pmatrix} 5 & -1 \\ -1 & 10 \end{pmatrix} \begin{pmatrix} c \\ -1 \end{pmatrix} = 4c - 9.$$

$$\text{So } c = \frac{9}{4}.$$

2. For X in 1, find the variance of its first principal component. (20 points)

The variance of the first principal component is the first (the largest) eigenvalue of Σ .

$$0 = |\Sigma - \lambda I| = \begin{vmatrix} 5 - \lambda & -1 \\ -1 & 10 - \lambda \end{vmatrix} = (\lambda - 5)(\lambda - 10) - 1 = \lambda^2 - 15\lambda + 49.$$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{15 + \sqrt{29}}{2}.$$

Thus the variance of the first principal component is $\frac{15 + \sqrt{29}}{2} \approx 10.1926$.

3. Write the SAS code including data step in order to get the SAS output with the result for 2.

(20 points)

```
data a (type='cov');
  _TYPE_='COV';
  input _NAME_ $ X1 X2;
datalines;
  X1 5 -1
  X2 -1 10
  ;
proc princomp cov;
  var X1 X2;
run;
```

4. Find all missing values in this SAS output and fill them in the blanks.

(30 points)

Eigenvalues of the Correlation Matrix

Eigenvalue	Difference	Proportion	Cumulative
1.80	1.10	0.600	0.60
0.70	0.20	0.233	0.833
0.50		0.167	1.000