Equilibrium Configurations for a Floating Drop

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1 Introduction

A variational formulation for the equilibrium configuration of three imiscible fluids in a closed container acted upon by a conservative force field was first given by L.A. Slobozhanin [11]. We will give a version of his argument using perterbations that are graphs and where the joining condition on the contact curve is derived variationally. An outline of the paper follows the derivation.

Consider three imiscible fluids in a vertical cylindrical container with gravity acting downwards. The fluids, labeled with subscripts 0,1,2, occupy regions $\Omega_0, \Omega_1, \Omega_2$. Their volumes V_0, V_1, V_2 are prescribed. Their densities are ρ_0, ρ_1, ρ_2 with $\rho_0 < \rho_1 < \rho_2$. The three interfaces between fluids S_{01}, S_{12}, S_{02} will have heights u, v, w, respectively, and are not assumed to be graphs. We suppose that a drop of fluid 1 is formed between fluids 0 and 2, above and below the drop, that all three surfaces meet along a curve Γ and that the interface S_{02} extends from Γ to the vertical wall of the cylinder. See Fig 1. The mean curvatures of the three interfaces will be denoted by H_u, H_v, H_w, with respect to an upward pointing normal, as indicated in Fig. 1. The prescribed surface tensions for the interfaces S_{01}, S_{12}, S_{02} are $\sigma_{01}, \sigma_{12}, \sigma_{02}$ and the surface tension between fluid 0 (1) and the cylinder is σ_{03} (σ_{23}). Let $|S_{ij}|$ be the area of S_{ij} for $j < 3$ of the area between fluid i and the container when $i = 0, 2$ and $j = 3$. The potential energy is then

$$E = \sum \sigma_{ij}|S_{ij}| + \sum \Phi_i$$

where $\Phi_i = \rho_i g \int\int\int_{\Omega_i} z dV$ is the gravitational potential energy of the fluid i, where z is a vertical coordinate and g is the acceleration due to gravity. The two fluid case is treated in the monograph of R. Finn [1]. Let x, y be coordinates in the horizontal plane. We derive

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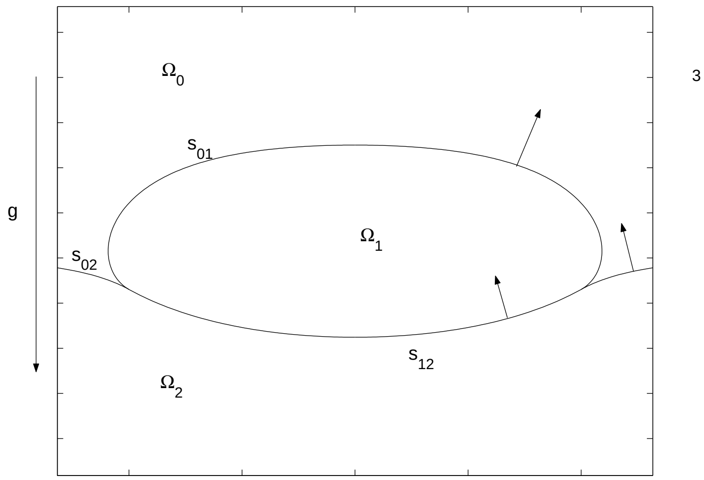

Figure 1: Drop configuration.

conditions for an equilibrium configuration by requiring that the energy E be stationary under perturbations which do not change any of the prescribed volumes. The argument proceeds in four steps.

1. Near an interior point p on S_{01} introduce local coordinates $\bar{x}, \bar{y}, \bar{z}$ where the $\bar{x}\bar{y}$-plane is tangent to S_{01} at p and $\bar{z}$ is in the upward direction. See Fig. 2.

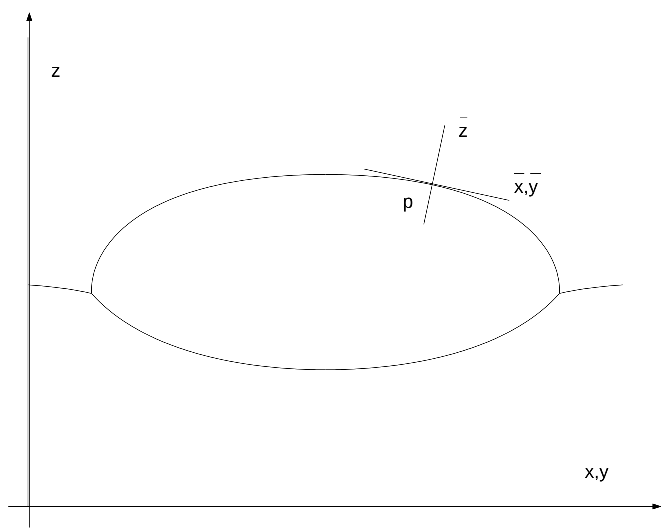

Figure 2: Horizontal and vertical coordinates and local coordinates.

Near p, S_{01} is a graph, $\bar{z} = \bar{u}(\bar{x}, \bar{y})$ for $(\bar{x}, \bar{y}) \in B$, B a ball centered at p. The vertical height is denoted u, so that $u = u(\bar{x}, \bar{y})$. Consider a perterbation $\eta \in C_0^2(B)$ with

$\int\int_B \eta dA = 0$. We have

$$\begin{aligned}\delta E &= \int\int_B [\sigma_{01} T\bar{u} \cdot \nabla\eta + (\rho_1 - \rho_0) g u \eta] dA \\ &= \int\int_B [-\sigma_{01} \nabla \cdot (T\bar{u}) + (\rho_1 - \rho_0) g u] \eta dA\end{aligned}$$

where $T\bar{u} = \nabla\bar{u}/\sqrt{1 + |\nabla\bar{u}|^2}$, $\nabla \cdot (T\bar{u}) = 2H_u$. So that $\delta E = 0$ implies $-2\sigma_{01} H_u + (\rho_1 - \rho_0) g u = \lambda_1$, λ_1 a constant. This condition holds on all of S_{01} with the same constant λ_1 since we assume that S_{01} is connected. Similarly,

$$\begin{aligned}-2\sigma_{12} H_v + (\rho_2 - \rho_1) g v &= \lambda_2 \quad \text{on } S_{12} \\ -2\sigma_{02} H_w + (\rho_2 - \rho_0) g w &= \lambda_3 \quad \text{on } S_{02}\end{aligned}$$

where λ_2, λ_3 are constants.

2. Let $p \in \Gamma$, where Γ is the contact curve where the three interfaces meet. Choose a local coordinate system so that near p each surface is a graph:

$$\begin{cases} \bar{z} = \bar{u}(\bar{x}, \bar{y}) & \text{in } B_1,\ \bar{z} = \bar{w}(\bar{x}, \bar{y}) \text{ in } B_2 \\ \bar{z} = \bar{v}(\bar{x}, \bar{y}) \end{cases}$$

where $B = B_1 \cup B_2$, B is a ball about p. Let Γ' be the projection of Γ in the $\bar{x}\bar{y}$-plane and let $\vec{\nu}$ be the unit normal on Γ' pointing out of B_1. See Fig 3. The argument

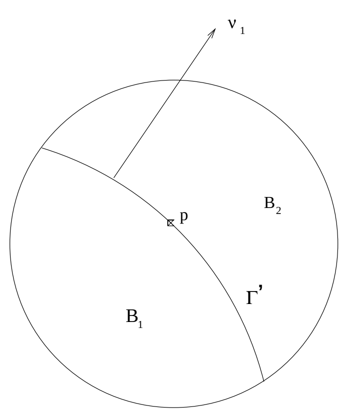

Figure 3: Local coordinate domains.

is similar if instead of $\bar{z} = \bar{v}(\bar{x}, \bar{y})$ in B_1 we have $\bar{z} = \bar{v}(\bar{x}, \bar{y})$ in B_2. Now consider a perturbation $\eta \in C_0^2(B)$ with $\int\int_{B_1} \eta dA = 0$, $\int\int_{B_2} \eta dA = 0$ so that the prescribed volumes are interchanged. We have

$$\delta E = \int_{\Gamma'} [\sigma_{01} T\bar{u} + \sigma_{12} T\bar{v} - \sigma_{02} T\bar{w}] \cdot \vec{\nu}_1 \eta ds$$

The upward normal on S_{01} is $\vec{n}_u = (-T\bar{u}, 1/\sqrt{1+|\nabla \bar{u}|^2})$ near p, and similarly for S_{12} and S_{02}. Let $\vec{\nu} = (\vec{\nu}_1, 0)$. $\delta E = 0$ implies $(\sigma_{01}\vec{n}_u + \sigma_{12}\vec{n}_v - \sigma_{02}\vec{n}_w) \cdot \vec{\nu} = 0$ at p. The vectors $\vec{n}_u, \vec{n}_v, \vec{n}_w, \vec{\nu}$ are coplanar since they are all orthogonal to Γ at p. If we rotate our local coordinate system slightly around the tangent direction to Γ at p we obtain the same condition with a $\vec{\nu}^*$ close to $\vec{\nu}$ but not a multiple of $\vec{\nu}$. This implies $\sigma_{01}\vec{n}_u + \sigma_{12}\vec{n}_v - \sigma_{02}\vec{n}_w = \vec{0}$ at p and so this condition holds on Γ.

An alternative form of this condition is $\sigma_{01}\vec{e}_u + \sigma_{12}\vec{e}_v - \sigma_{02}\vec{e}_w = \vec{0}$ on Γ where $\vec{e}_u$ is a unit vector on Γ, orthogonal to Γ, tangent to S_{01} and in the direction of S_{01}; $\vec{e}_v$ and $\vec{e}_w$ are defined similarly. See Fig 4. The alternate form follows since $\vec{e}_u, \vec{e}_v$, and $\vec{e}_w$ are

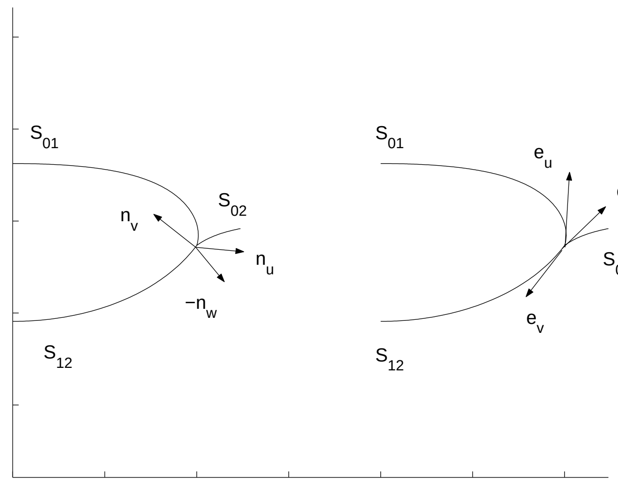

Figure 4: Normal vectors and tangent vectors on Γ.

obtained by rotating $\vec{n}_u, \vec{n}_v$, and $-\vec{n}_w$ counterclockwise about Γ. This known condition is interpreted as force balance along Γ, see [6] for example. Expressed in terms of contact angles this becomes

$$\frac{\sin \gamma_{01}}{\sigma_{01}} = \frac{\sin \gamma_{02}}{\sigma_{02}} = \frac{\sin \gamma_{12}}{\sigma_{12}}$$

where the contact angles along Γ are indicated in Fig 5. γ_{02} is the contact angle between S_{01} and S_{02}, etc.

3. Next perturb all the surfaces simultaneously with $\eta_{01}, \eta_{12}, \eta_{02}, \eta_{ij} \in C_0^2(B_{ij})$, where $\iint_{B_{01}} \eta_{01} dA = \iint_{B_{12}} \eta_{12} dA = -\iint_{B_{02}} \eta_{02} dA \neq 0$, so that the volumes of the fluids are unchanged. $\delta E = 0$ implies $\int_{B_{01}} \lambda_1 \eta_{01} dA = \int_{B_{12}} \lambda_1 \eta_{12} dA = -\int_{B_{02}} \lambda_1 \eta_{02} dA = 0$, giving $\lambda_1 + \lambda_2 - \lambda_3 = 0$.

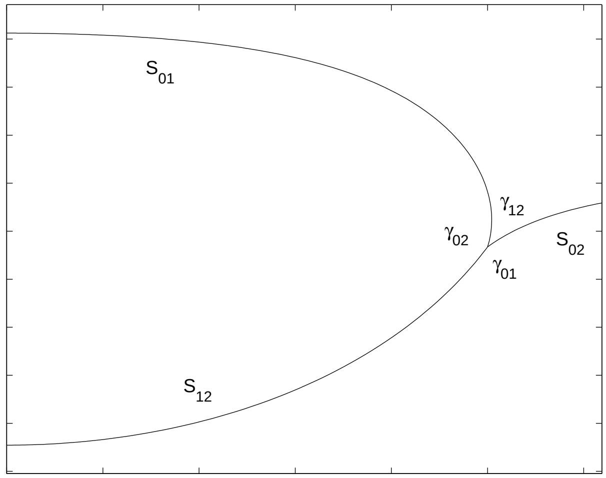

Figure 5: Contact angles along Γ.

4. Finally, the surface S_{02} meets the wall of the cylinder along the curve Γ_0. Let $p \in \Gamma_0$. Near p, $z = w(x, y)$ for $(x, y) \in B_1$ where B_1 is a portion of a ball B about p. Let Γ_0' be the projection of Γ_0 onto the xy-plane. See Fig. 6. Let $\vec{\nu}_1$ be the unit normal on

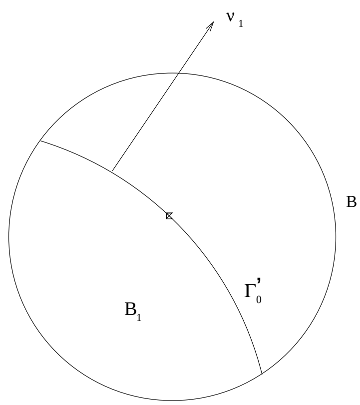

Figure 6: The ball B and domain B_1.

Γ_0' pointing out of B_1. Now perturb with $\eta \in C_0^2(B)$, $\iint_{B_1} \eta dA = 0$.

$$\delta E = \int_{\Gamma_0'} [\sigma_{02} T w \cdot \vec{\nu}_1 + (\sigma_{23} - \sigma_{03})] \eta ds. \tag{1}$$

Since η can be arbitrary on Γ_0', $\delta E = 0$ implies $\sigma_{02} T w \cdot \vec{\nu}_1 = \sigma_{03} - \sigma_{23}$ or $\cos \gamma_0 = (\sigma_{03} - \sigma_{23})/\sigma_{02}$ where γ_0 is the contact angle in fluid 2 at Γ_0. This last argument is completely standard, see [1].

Next consider a finite drop between infinite media. This may be considered a limiting case of the finite container. Suppose $w \to 0$ as $r \to \infty$ $(r = \sqrt{x^2 + y^2})$. Then we must have $\lambda_3 = 0$. Let $\lambda = \lambda_1$, $Mu = 2H_v$, $Mw = 2H_w$, $\kappa_{ij} = (\rho_j - \rho_i)g/\sigma_{ij}$ (capillary constants). We have

$$\begin{cases} Mu = k_{01}u + \lambda/\sigma_{01} \\ Mv = \kappa_{12}v - \lambda/\sigma_{12} \\ Mw = \kappa_{02}w \end{cases} \tag{2}$$

$$u, v, w \text{ meet at } \Gamma \text{ and satisfy the force balance condition.} \tag{3}$$

In this paper we will consider axisymmetric solutions to (2)-(3). In this case the contact curve Γ will be a circle of radius $\bar{r}$, which we call the radius of the drop. We do not assume that the surfaces are graphs. The assumption on the densities gives that $\kappa_{01}, \kappa_{12}, \kappa_{02} > 0$ so that u and v are symmetric interior capillary surfaces and w is a symmetric exterior capillary surface. Let $\bar{\psi}$ denote the inclination angle of v at $\bar{r}$.

The orginization of the paper is as follows. In section 2 we give some preliminary results on symmetric interior and exterior capillary surfaces. These were first studied by Johnson and Perko [5]. From results of Finn [1] we deduce the existence and uniqueness of an interior capillary surface which has a given inclination angle at a given radius (Theorem 2.1). Extending the work of Vogel [13] we obtain the extistence and uniqueness of an exterior capillary surface which has a given inclination angle at a given radius (Theorem 2.2). The uniqueness part of Theorem 2.2 requires a volume comparison argument. In Section 3 we prove the existence of a drop of a given radius by a continuity method (Theorem 3.1). When $0 \leq \gamma_{02} \leq \pi/2$ we show that the drop of a given radius is unique and that for every prescribed volume there is a drop with that volume (Theorem 3.2). The uniqueness comes from monotonicity properties of nonparametric interior and exterior capillary surfaces due to Finn and Siegel, see [1] and [10]. The existence and uniqueness of a drop of a given radius in this case was initially presented by S.T. Gibbs [2]. Then we give an asymptotic result for small drops. The inclination angle $\bar{\psi}$ tends to a specific limit as $\bar{r}$ tends to 0. In Section 4 we consider the floating "bubble": $\rho_0 = \rho_1 < \rho_2$, $\sigma_{01}/2 = \sigma_{02} = \sigma_{12}$. In this case $\gamma_{02} = \pi, \gamma_{01} = 0, \gamma_{12} = \pi$. We prove that there is a unique bubble of given radius and that for every prescribed volume there is a bubble with that volume (Theorems 4.1 and 4.3).

The inclination angle $\bar{\psi}$ tends to π as $\bar{r}$ tends to 0 (Theorem 4.2). Open questions are stated in Section 5.

Hartland and Hartley [3] have calculated drops in the special case $\rho_1 = \rho_2$, $\sigma_{01} + \sigma_{02} = \sigma_{12}$ or $\sigma_{01} = \sigma_{02} = \sigma_{12}/2$. The second case is transformable to the floating bubble case considered in section 4 [$(-u, -v, -w)$ satisfies the floating bubble equations]. Floating drops are studied experimentally in the papers of Princen [7, 8, 9].

2 Preliminaries

We need two results on existence and uniqueness of capillary surfaces. They may also be interpreted in terms of sessile drops and liquid bridges. Consider first an axisymmetric capillary surface with center height u_0. Following Finn [1] we can represent this surface using the tangent angle ψ as a parameter, and the surface is given by solving the differential equations

$$\frac{dr}{d\psi} = \frac{r\cos\psi}{\kappa r u - \sin\psi}, \quad \frac{du}{d\psi} = \frac{r\sin\psi}{\kappa r u - \sin\psi} \tag{4}$$

with $r(0) = 0, u(0) = u_0$. The surface can only be thought of as a graph $u = u(r)$ for $\psi \in [0, \pi/2]$, but it is essential for us to use this representation for $\psi \in [0, \pi]$. If the surface is turned over it represents the profile of a sessile drop and this paramtereization was used [1, chapter 3] to prove the existence of a unique sessile drop for every prescribed contact angle and volume.

Theorem 2.1 *For every $\bar{r} > 0$ and $\bar{\psi} \in (0, \pi]$ there is a unique solution of (4) such that $r(\bar{\psi}) = \bar{r}$.*

Proof. Since solutions of (4) may be thought of as $r = r(\psi; u_0), u = u(\psi; r_0)$ we must show that there is a unique $u_0 > 0$ such that $r(\bar{\psi}; u_0) = \bar{r}$.

It has been proven in [1, theorem 3.2] that $\partial r/\partial u_0 < 0$.

We introduce the notation $R = r(\pi/2; u_0), a = r(\pi; u_0)$. Since $R < 2/\kappa u_0$ we have $R \to 0$ as $u_0 \to \infty$. From [1, 2.63], $R \to \infty$ as $u_0 \to 0$. We can now deduce that$r(\bar{\psi}; u_0) \to 0$ as $u_0 \to \infty$ and $r(\bar{\psi}; u_0) \to \infty$ as $u_0 \to 0$. The first statement follows from $r(\bar{\psi}; u_0) \leq R$. For the second, if $\psi \leq \pi/2$ we can use [1, 2.60]. For $\psi > \pi/2$, we use $a \leq r(\bar{\psi}; u_0) \leq R$ and $a \to \infty$ as $u_0 \to 0$ which follows from [1, 3.91].

The theorem follows from the monotonicity of r with respect to u_0. ■

For the exterior surfaces there is an analgous representation [13]

$$\frac{du}{d\phi} = \frac{-r\sin\phi}{\kappa r u + \sin\phi}, \quad \frac{dr}{d\phi} = \frac{-r\cos\phi}{\kappa r u + \sin\phi} \tag{5}$$

where ϕ is the tangent angle, taken as acute on the "top" of the curve. Johnson and Perko [5] have shown that for each $\sigma > 0$ there is a unique solution with vertical tangent at σ, $r \to \infty, u \to 0$ as $\phi \to \pi$, with $u = u(r)$. The above representation allows us to convienently extend this curve to $[0, \pi)$. We denote by $T(\sigma)$ the height at σ, ie. $T(\sigma) = u(\pi/2)$. The parameter σ plays a role for exterior surfaces analgous to that u_0 plays for interior ones. It has been shown by Siegel [10] that $T(\sigma)$ is strictly increasing in σ, Vogel [13] has shown that $r(0; \sigma) \to 0$ as $\sigma \to 0$, and Turkingston [12] has shown that $T(\sigma) \sim \sigma \ln(1/\sigma)$ as $\sigma \to 0$.

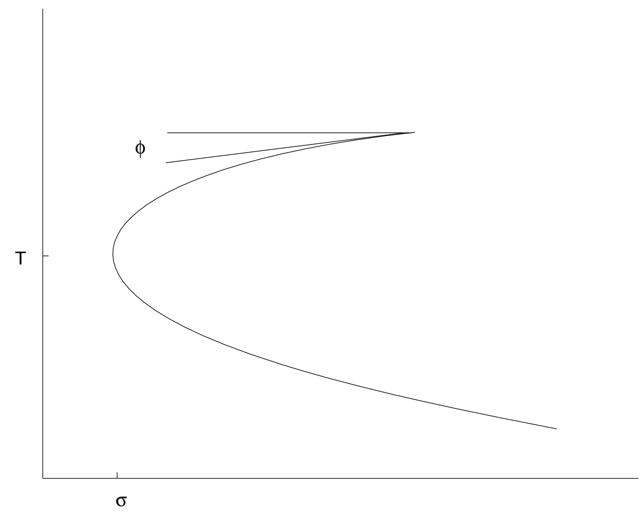

Figure 7: Exterior surface with tangent angle ϕ.

Theorem 2.2 *For every $\bar{r} > 0$ and $\bar{\phi} \in [0, \pi)$ there is a unique solution of (5) with $r(\pi/2; \sigma) = \sigma, u(\pi/2; \sigma) = T(\sigma)$ such that $r(\bar{\phi}; \sigma) = \bar{r}$.*

For $\bar{\phi} \in [\pi/2, \pi]$ this was proven in [10]. For $\hat{\phi} \in [0, \pi/2)$ we need only observe $\sigma \leq r(\hat{\phi}; \sigma) \leq r(0; \sigma)$ and use continuity of solutions of (5) with respect to intial conditions to deduce existence.

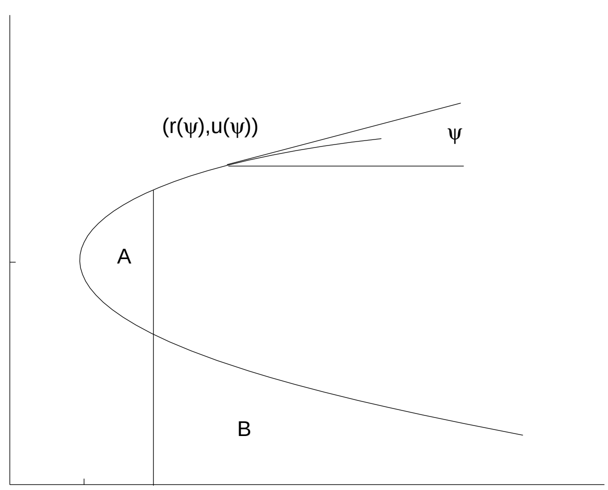

Figure 8: A and B regions.

The proof of uniqueness is considerably more involved. We need the following lemmas due to Vogel [13].

Lemma 2.1 *Let $\Gamma = (r(\phi), u(\phi))$ be a particular profile curve. Pick $\phi_0 \in [0, \pi)$, and let $r_0 = r(\phi_0)$. Let A be the solid obtained by rotating the region bounded by $r = r_0$ and Γ and let B be the solid obtained by rotating the unbounded region between Γ and the r-axis from $r = r_0$ to $r = +\infty$. Then*

$$|B| - |A| = 2\pi r_0 \sin\phi_0 \tag{6}$$

Lemma 2.2 *Let Γ_1 and Γ_2 be profile curves as above. If for some $\phi_0 \in [0, \pi)$ we have $u(\phi_0)_1 = u(\phi_0)_2$, then $\Gamma_1 \equiv \Gamma_2$.*

Lemma 2.3 *No two distinct profile curves can cross twice.*

Proof. (of uniqueness): We need to show that if Γ_1 and Γ_2 are two profile curves as above with $r(\phi_0)_1 = r(\phi_0), \phi_0 \in [0, \pi/2)$, that Γ_1 and Γ_2 coincide.

Suppose that there are two such curves. Let σ_1, σ_2 be the radii at which they are vertical. We may assume that $u(\phi_0)_1 > u(\phi_0)_2$. Suppose that ϕ_0 is the largest value of ϕ such that $r(\phi)_1 = r(\phi)_2 = \rho_0$. (These are the leftmost intersection points.) ie. ρ_0 is the smallest radius. Let A_1, B_1, A_2, B_2 be the volumes in Lemma 2.1 for these two curves.

We consider the intersection points of these curves with $r = \rho_0$. Denote by $\alpha_U = u(\phi_0)_1, \beta_U = u(\phi_0)_2$ the upper intersection heights ($\alpha_U > \beta_U$) and α_L, β_L the lower intersection heights.

Case 1. $\alpha_L \leq \beta_L$. We have then $\alpha_U > \beta_U > \beta_L \geq \alpha_L$. Since Γ_1 and Γ_2 cannot cross more than once they certainly cannot cross trice for $r \leq \rho_0$ and, hence, they cannot cross at all, so that Γ_1 lies entirely inside the region bounded by Γ_2 and $r = \rho_0$, for $r = \rho_0$. It follows that $\sigma_1 < \sigma_2$ and $T(\sigma_1) < T(\sigma_2)$, which implies there is a $\phi \in (\phi_0, \pi/2)$ where $u(\phi)_1 = u(\phi)_2$ contradicting Lemma 2.2.

Case 2. $\alpha_L > \beta_L$. As above $T(\sigma_1) \leq T(\sigma_2)$ implies that there are two intersection points contradicting Lemma 2.3, so $T(\sigma_1) > T(\sigma_@)$ and $\sigma_1 > \sigma_2$.

We begin by showing that $|B_1| > |B_2|$. First consider what happens when $\alpha_L < \beta_U$. Then Γ_1 and Γ_2 must cross somewhere above α_L. Thus they cannot cross again below α_L and since $\alpha_L > \beta_L$ this implies that the lower portion of Γ_1 for $r > \rho_0$ lies completely above the corresponding portion of Γ_2. Thus it follows that $|B_1| > |B_2|$. Now suppose $\alpha_L > \beta_U$. Then the part of Γ_1 for $r < \rho_0$ lies entirely above the corresponding part of Γ_2. Thus at $u = \beta_U$ Γ_1 has inclination angle greater than $\pi/2$ while Γ_2 has inclination angle less than $\pi/2$. Also, at $r = \rho_0$ the lower part of Γ_1 is above the lower part of Γ_2. We wish to show that Γ_1 remains above Γ_2 for all $r > \rho_0$. If this were not the case, the two curves would have to cross somewhere below $u = \beta_U$. At the point of crossing , the inclination angle of Γ_1 would have to be less than that of Γ_2. Since at $u = \beta_u$ this inequality was reversed, there must be a value of u between β_U and the value of u at the point of crossing at which both curves have the same inclination angle. But by Lemma 2.2, this would imply that Γ_1 and

Γ_2 are identical. hence they cannot cross, and the lower part of Γ_1 lies above the lower part of Γ_2 for all $r > \rho_0$. Thus $|B_1| > |B_2|$.

Now we wish to show that $|A_1| < |A_2|$. Let Γ_1' be the rigid translation of Γ_1 downward by $\alpha_U - \beta_U$. Then we wish to show that Γ_1' for $r \leq \rho_0$ is contained in the region bounded by Γ_2 and $r = \rho_0$. We see that Γ_1' and Γ_2 are tangent at β_U. Locally they are both functions $u(r)$ so we consider the second derivative. We compute (using the above differential equations):

$$\frac{d^2u}{dr^2} = \frac{1}{\cos^2\phi}\frac{ru + \cos\phi}{-r\cos\phi} \tag{7}$$

Since ϕ and r are equal for both curves but u is greater for Γ_1' (the differential equation used in evaluating derivatives for Γ_1' uses u values on Γ_1), it follows that on Γ_1' we have a smaller (negative) value for d^2u/dr^2, and, hence, for r slightly smaller than ρ_0, Γ_1' lies below Γ_2 and has a greater inclination angle.

So we know Γ_1' starts "inside" Γ_2, and now we must ask if can ever leave. Suppose Γ_1' leaves on the upper portion of Γ_2. Then at the point of leaving Γ_1' has a smaller inclination angle than Γ_2 and thus for some r between ρ_0 and the radius at the point of leaving, Γ_1' and Γ_2 have the same inclination angle. Thus Γ_1 and Γ_2 have the same inclination angle at that radius. But since this radius is smaller than ρ_0, this contradicts our choice of ρ_0 as the smallest radius at which the two curves have the same inclination angle. Thus Γ_1' cannot leave across the upper portion of Γ_2.

Now it remains to consider whether Γ_1' can leave across the lower portion of Γ_2 for $r \leq \rho_0$. If it did, then it could not return since if it did we would again have a radius smaller than ρ_0 at which the curves had equal inclination angles by the same argument as above. So a necessary condition for Γ_1' to "escape" is $\alpha_U - \alpha_L > \beta_U - \beta_L$. We prove that this cannot happen. Observe that $\alpha_U - \alpha_L = (\alpha_U - T(\sigma_1)) + (T(\sigma_1) - \alpha_L)$ and $\beta_U - \beta_L = (\beta_U - T(\sigma_2)) + (T(\sigma_2) - \beta_L)$. So first we compute

$$\alpha_U - T(\sigma_1) = \int_{\phi_0}^{\pi/2} \frac{r_1 \sin\phi}{r_1 u_1 + \sin\phi} d\phi \tag{8}$$

$$\beta_U - T(\sigma_2) = \int_{\rho_0}^{\pi/2} \frac{r_2 \sin\phi}{r_2 u_2 + \sin\phi} d\phi \tag{9}$$

At ϕ_0, we see that $\frac{\rho_0 \sin\phi_0}{\rho_0 u_2 + \sin\phi_0}$, the integrand corresponding to Γ_2, is greater than $\frac{\rho_0 \sin\phi_0}{\rho_0 u_1 + \sin\phi_0}$, the integrand corresponding to Γ_1. We claim that this inequality holds through the interval of integration. If it did not there would be some ϕ at which $\frac{r_1 \sin\phi}{r_1 u_1 + \sin\phi} = \frac{r_2 \sin\phi}{r_2 u_2 + \sin\phi}$ which is equivalent to $(u_2 - u_1) r_1 r_2 = (r_2 - r_1)\sin\phi$. But the left side of this last equation is negative throughout the interval while the right side is positive. Thus the integrands can never be equal and the previous inequality holds throughtout the interval. This shows that $\beta_U - T(\sigma_2) > \alpha_U - T(\sigma_1)$.

Now we compute

$$T(\sigma_1) - \alpha_L = \int_{\pi/2}^{\theta_1} \frac{r_1 \sin\phi}{r_1 u_1 + \sin\phi} d\phi \tag{10}$$

$$T(\sigma_2) - \beta_L = \int_{\pi/2}^{\theta_2} \frac{r_2 \sin\phi}{r_2 u_2 + \sin\phi} d\phi \tag{11}$$

where θ_1 and θ_2 are the inclination angles at which the lower portions of Γ_1 and Γ_2 respectively intersect $r = \rho_0$. Since $\sigma_1 > \sigma_2$ it follows that the inclination angle of Γ_1 at $\sigma_1, \pi/2$, is less than the corresponding angle for Γ_2. Since ρ_0 is the smallest radius at which the two can have the same inclination angle, this continues to be true for all $r < \rho_0$ and thus that $\theta_2 \geq \theta_1$. Further, the same argument as in the previous set of integrals shows that the integrand corresponding to Γ_2 is again greater that that corresponding to Γ_1. Since both integrands are positive and the interval of integration corresponding to Γ_2 contains the interval corresponding to Γ_1 it follows that $T(\sigma_2) - \beta_L > T(\sigma_1) - \alpha_L$.

Combining the above results shows that $\alpha_U - \alpha_L > \beta_U - \beta_L$. As mentioned, this proves that Γ_1' cannot "escape" through the lower portion of Γ_2 and thus cannot escape at all. Since Γ_1' for $r < \rho_0$ is contained in the region bounded by Γ_2 and $r = \rho_0$, it follows that the volume obtained by rotating that portion of Γ_2 is greater than the corresponding volume obtained from Γ_1'. But Γ_1' gives the same volume as Γ_1 since they merely differ by a u translation. This we have shown that $|A_1| < |A_2|$.

The above results yield the inequality $|B_1| - |A_1| > |B_2| - |A_2|$. However, lemma 2.1 tells us that $|B_1| - |A_1| = 2\pi\rho_0 \sin\phi_0 = |B_2| - |A_2|$. Thus we have our contradiction, and then the theorem is proven. ■

3 Existence of a Floating Drop

We consider a finite volume of fluid of density ρ_1 laying on an infinite reservoir of a fluid of density ρ_2 below another infinite reservoir of a fluid of density ρ_0. We will assume that $\rho_0 < \rho_1 < \rho_2$.

We will take as given the surface tensions $\sigma_{01}, \sigma_{02}, \sigma_{12}$ at a contact line between the three fluids, the subscripts indicate

that the force acts along the interface between the indicated fluids. The corresponding capillary constants are defined by

$$\begin{aligned} \kappa_{01} &= (\rho_1 - \rho_0)g/\sigma_{01}, \\ \kappa_{02} &= (\rho_2 - \rho_0)g/\sigma_{02}, \\ \kappa_{12} &= (\rho_2 - \rho_1)g/\sigma_{12}. \end{aligned} \tag{12}$$

As we have seen, assuming that the three surfaces are axisymmetric, minimization of energy implies that they satisfy the equations

$$\begin{aligned} Mu &= \kappa_{01}u + \lambda/\sigma_{01}, \\ Mv &= \kappa_{12}v - \lambda/\sigma_{12}, \\ Mw &= \kappa_{02}w \end{aligned}$$

where λ is a constant which must be determined as a part of the solution of the problem.

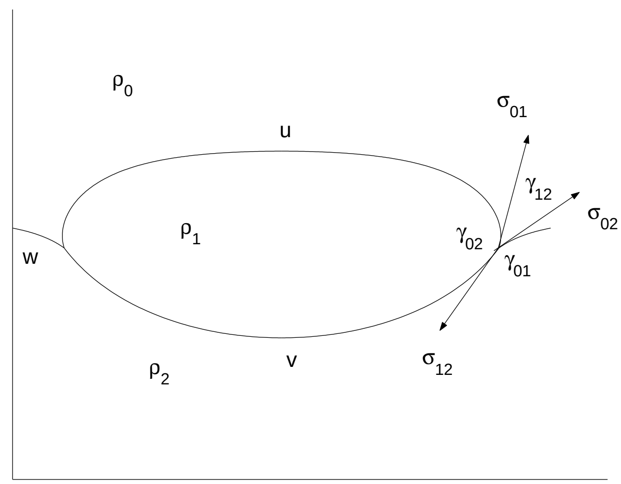

Figure 9: Contact angles and surface tensions.

By force balance (cf. Section 1) $0 \leq \gamma_{01}, \gamma_{02}, \gamma_{12} \leq \pi$, and

$$\begin{aligned} \gamma_{02} &= \pi - \arccos(\frac{\sigma_{01}^2 + \sigma_{12}^2 - \sigma_{02}^2}{2\sigma_{01}\sigma_{12}}), \\ \gamma_{01} &= \pi - \arccos(\frac{\sigma_{12}^2 + \sigma_{02}^2 - \sigma_{01}^2}{2\sigma_{12}\sigma_{02}}), \end{aligned}$$

(of course $\gamma_{12} = 2\pi - \gamma_{01} - \gamma_{02}$), or, equivalently,

$$\frac{\sin \gamma_{01}}{\sigma_{01}} = \frac{\sin \gamma_{02}}{\sigma_{02}} = \frac{\sin \gamma_{12}}{\sigma_{12}}.$$

We note that $\sigma_{01}\kappa_{01} + \sigma_{12}\kappa_{12} = \sigma_{02}\kappa_{02}$. We will fix the radius $\bar{r}$ at which the three surfaces meet, and let $\bar{\psi}$ be the angle that the tangent to v at $\bar{r}$ makes with the horizontal.

We must have $0 < \bar{\psi} < \gamma_{02}$. If this were not the case then the drop would be "tipped" up too far and the surface u would be inclined instead of beginning at a zero slope and decreasing. In fact if $\bar{\psi} = \gamma_{02}$, then the fact that the supplement of the sum of γ_{02} and

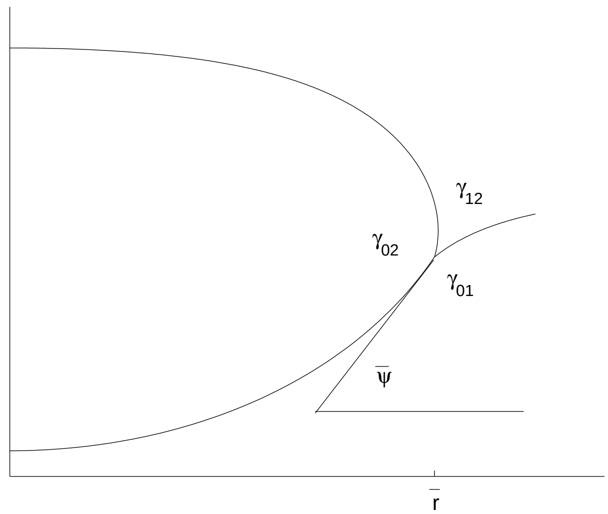

Figure 10: Inclination angle at the prescribed radius.

the complement of $\bar{\psi}$ is equal to $\pi/2 - \gamma_{02} + \bar{\psi}$ implies this angle is $\pi/2$. Stated differently, the slope at $\bar{r}$ is zero , and thus the surface u is flat, and any larger angle $\bar{\psi}$ should be excluded from our consideration. This also gives us our first relation between the different boundary conditions. If we have $\bar{\psi}$ for the surface v, then for the surface u $\gamma_{02} - \bar{\psi}$ is the angle downward from horizontal. There are several possibilities for the outer surface: w slopes down to the intersection point, w slopes up to the intersection point, and one case of each where w is curled over. The last two cases may seem physically unlikely, but are included for completeness.

There are further geometrical constraints. They do not play an explicit role in what follows, but they are included for completeness. We have two cases for $0 \leq \gamma_{02} < \pi/2$: if $\gamma_{02} \leq \gamma_{12}$ then w slopes down, and if not then w slopes up. However if we consider $\gamma_{02} > \pi/2$, the situation is more complicated. We break this into two cases. If $\bar{\psi} \leq \pi/2$, the interface is below the middle of the drop, and if $\bar{\psi} > \pi/2$ the interface is above the middle of the drop. There are three configurations contained in each case. First we consider $\bar{\psi} \leq \pi/2$. If $0 \leq \bar{\psi} \leq (\pi/2 - \gamma_{01})$ then w is folded over and sloping down. If $(\pi/2 - \gamma_{01}) \leq \bar{\psi} \leq (\pi - \gamma_{01})$ then w is merely sloping down. If $(\pi - \gamma_{01}) \leq \bar{\psi} \leq \gamma_{02}$ then w is sloping up. For the second case, we have $\bar{\psi} > \pi/2$. If $0 \leq \bar{\psi} \leq (\pi - \gamma_{01})$ then w is sloping down. If $(\pi - \gamma_{01}) \leq \bar{\psi} \leq (3\pi/2 - \gamma_{01})$, then w slopes up. If $(3\pi/2 - \gamma_{01}) \leq \bar{\psi} \leq (\pi - \gamma_{01})$ then w is folded over, sloping up.

We can match the top and the bottom surfaces, u and v, at $\bar{r}$ by eliminating the Lagrange multiplier λ. Define U and V by $u = -U - \lambda/\kappa_{01}\sigma_{01}$ and $v = V + \lambda/\kappa_{12}\sigma_{01}$.

Then $MU = \kappa_{01}U$ and $MV = \kappa_{12}V$, and $u(\bar{r}) = v(\bar{r})$ implies that

$$\lambda = -[U(\bar{r}) + V(\bar{r})]\frac{\kappa_{01}\sigma_{01}\kappa_{12}\sigma_{12}}{\kappa_{01}\sigma_{01} + \kappa_{12}\sigma_{12}}.$$

If we put this back into the equations that defined U and V we get

$$u(\bar{r}) = v(\bar{r}) = \frac{\kappa_{12}\sigma_{12}V(\bar{r}) - \kappa_{01}\sigma_{01}U(\bar{r})}{\kappa_{01}\sigma_{01} + \kappa_{12}\sigma_{12}}.$$

The surfaces U and V are capillary surfaces or inverted sessile drops as described in Theorem 2.1. V makes the angle $\bar{\psi}$ at $\bar{r}$, and U makes the angle $\gamma_{02} - \bar{\psi}$.

We write the difference of u and w as

$$F(\bar{\psi}) = \frac{\kappa_{12}\sigma_{12}V(\bar{r}) - \kappa_{01}\sigma_{01}U(\bar{r})}{\kappa_{01}\sigma_{01} + \kappa_{12}\sigma_{12}} - w(\bar{r}) \tag{13}$$

and vary $\bar{\psi}$.

By looking at limiting cases we see that $F(0) < 0$, since then $V \equiv 0$, $U > 0$, and $w \geq 0$. Similarly, $F(\gamma_{02}) > 0$, since then $V > 0$, $U \equiv 0$, and $w \leq 0$. (we have used here that fact that w is zero at infinity.)

We have proven

Theorem 3.1 *For each $\bar{r}$, there is a drop in which the three surfaces meet at radius $\bar{r}$.*

For $0 < \gamma_{02} < \pi/2$ all three surfaces u, v, w must be graphs. In this case we have

Theorem 3.2 *Suppose that $0 < \gamma_{02} < \pi/2$. Then there is a drop of volume $\mathcal{V}$ for every prescribed $\mathcal{V} > 0$.*

Proof. First we observe that the results of Siegel [10] show that $\partial F/\partial\bar{\psi} > 0$. Then there is a unique $\bar{\psi}(\bar{r})$ satisfying $F(\bar{\psi}) = 0$ for each $\bar{r} > 0$, and $\bar{\psi}(\bar{r})$ depends continuously on $\bar{r}$.

The volume can be written as the sum of volumes of two sessile drops [1, page 40] for the volume of a sessile drop

$$\begin{aligned}
\mathcal{V} &= \mathcal{V}_1 + \mathcal{V}_2 \\
\mathcal{V}_1 &= \pi\bar{r}(\bar{r}V(\bar{r}) - 2\sin\bar{\psi}/\kappa_{12}), \\
\mathcal{V}_2 &= \pi\bar{r}(\bar{r}U(\bar{r}) - 2\sin(\gamma_{02} - \bar{\psi})/\kappa_{01}),
\end{aligned}$$

we can estimate $\mathcal{V}$ from below by considering two cases, $\bar{\psi} \geq \gamma_{02}/2$, $\bar{\psi} < \gamma_{02}/2$. In the first case

$$\begin{aligned}
\mathcal{V}_1 &\geq \pi\bar{r}(\bar{r}V(\bar{r}, \gamma_{02}/2) - 2/\kappa_{12}) \\
&\geq \pi\bar{r}(\bar{r}\sqrt{\frac{2}{\kappa_{12}}(1 - \cos(\frac{\gamma_{02}}{2}))} - \frac{2}{\kappa_{12}})
\end{aligned}$$

where the second inequality follows from [10, theorem 7]. In the second case

$$\mathcal{V}_2 \geq \pi\bar{r}(\bar{r}\sqrt{\frac{2}{\kappa_{01}}(1-\cos(\frac{\gamma_{02}}{2}))} - \frac{2}{\kappa_{01}}).$$

Combining these

$$\mathcal{V} \geq \pi\bar{r}(\bar{r}\sqrt{\frac{2}{\kappa_1}(1-\cos(\frac{\gamma_{02}}{2}))} - \frac{2}{\kappa_2})$$

where $\kappa_1 = \max(\kappa_{01}, \kappa_{12}), \kappa_2 = \min(\kappa_{01}, \kappa_{12})$. It follows that $V \to \infty$ as $\bar{r} \to \infty$. Also $\mathcal{V} \leq 2\pi\bar{r}^2 V(\bar{r}, \pi/2)$, and $V(\bar{r}, \pi/2)$ is $\mathcal{O}(1/\bar{r})$ [1, p27], so $V = \mathcal{O}(\bar{r})$ as $\bar{r} \to 0$. The theorem follows. ■

An analogue of Theorem 3.1 can be given for a finite circular cylindrical container with an axisymetric drop lying on the symmetry axis. The role of w is taken over by a surface over an annular domain. The only property of w which was used in showing that $F(\bar{\psi})$ changes sign is that $w'(\bar{r}) \leq 0$ implies that $w(\bar{r}) \geq 0$ and $w'(\bar{r}) \geq 0$ implies that $w(\bar{r}) \leq 0$. Assuming that the contact angle at the container boundary is in $[0, \pi/2]$, this follows from the basic comparison theorem for capillary surfaces, [1, theorem 5.1], if w is compared with zero. The general case requires detailed properties of capillary surfaces over annular domains which will not be discussed here.

We now give some asymptotic results for small drops.

Lemma 3.1 *For a capillary surface V determined by $\bar{r}$ and $\bar{\psi}$ (cf. Theorem 2.1),*

$$V = \frac{2\sin\bar{\psi}}{\kappa\bar{r}} + \mathcal{O}(\bar{r}), \quad \text{as } \bar{r} \to 0,$$

uniformly for $\bar{\psi}$ bounded away from π.

Proof. Case 1. $\bar{\psi} \leq \pi/2$. This follows from [1, 2.42] and [10].

Case 2. $\pi/2 \leq \bar{\psi} \leq \pi - \epsilon$ for some $\epsilon > 0$. First

$$V = 2/\kappa R + \mathcal{O}(R) \quad \text{as } R \to 0$$

by [1, pg. 55]. Then

$$\bar{r} = R\sin\bar{\psi} + \mathcal{O}(R^3)$$

by [1, pg 52-53, 3.46, 3.53b, 3.54c]. Combining these gives the result. ■

Definition 3.1 *For $0 < \gamma_{02} < \pi$ define γ_c as the unique root of*

$$\frac{\sin\psi}{\sin(\gamma_{02}-\psi)} = \frac{\sigma_{01}}{\sigma_{12}}$$

in $(0, \gamma_{02})$.

This makes sense since $g(\psi) = \sin\psi/\sin(\gamma_{02} - \psi)$ is strictly increasing on $(0, \gamma_{02})$ and $g(0) = 0$, $\lim_{\psi\to\gamma_{02}-} g(\psi) = \infty$.

Theorem 3.3 *1. Suppose that $0 < \gamma_{02} < \pi$. Then for $\epsilon > 0$ there is an $r_0 > 0$ such that $F(\bar\psi) \neq 0$ on $[0, \gamma_c - \epsilon] \cup [\gamma_c + \epsilon, \pi]$ for $0 < r \leq r_0$.*

2. Suppose that $\gamma_{02} = \pi$. Then for $\epsilon > 0$ there is an $r_0 > 0$ such that $F(\bar\psi) \neq 0$ on $[\epsilon, \pi - \epsilon]$ for $0 < r \leq r_0$.

Proof. Lemma 3.1 implies that

$$V = \frac{2\sin\bar\psi}{\kappa_{12}\bar r} + \mathcal{O}(\bar r) \quad \text{as } \bar r \to 0$$

and

$$U = \frac{2\sin(\gamma_{02} - \bar\psi)}{\kappa_{01}\bar r} + \mathcal{O}(\bar r) \quad \text{as } \bar r \to 0.$$

Using results of Turkington [12] and Vogel [13] $w = \mathcal{O}(\bar r \ln \bar r)$ as $\bar r \to 0$, so

$$\begin{aligned} F(\bar\psi) &= \frac{\kappa_{12}\sigma_{12}\dfrac{2\sin\bar\psi}{\kappa_{12}\bar r} - \kappa_{01}\sigma_{01}\dfrac{2\sin(\gamma_{02} - \bar\psi)}{\kappa_{01}\bar r}}{\kappa_{01}\sigma_{01} + \kappa_{12}\sigma_{12}} + \mathcal{O}(\bar r \ln \bar r) \\ &= \frac{2}{\kappa_{01}\sigma_{01} + \kappa_{12}\sigma_{12}} \frac{1}{\bar r}(\sigma_{12}\sin\bar\psi - \sigma_{01}\sin(\gamma_{02} - \bar\psi)) + \mathcal{O}(\bar r \ln \bar r) \quad \text{as } \bar r \to 0 \end{aligned}$$

■

This theorem shows that as $\bar r$ tends to zero a $\bar\psi$ such that $F(\bar\psi) = 0$ tends to ψ_c in case 1 and either to 0 or π in case 2.

4 The Floating Bubble

We give a seperate treatment to a limiting case of the floating drop in which $\rho_0 = \rho_1$, $\sigma_{12} = \sigma_{02} := \sigma$ and $\sigma_{01} = 2\sigma$. We may imagine a bubble of the top fluid, eg. air, which is bounded on top by a thin film of the bottom fluid. The contact angles are $\gamma_{02} = \pi$, $\gamma_{01} = 0$, $\gamma_{12} = \pi$. The differential equations become

$$Mu = \lambda/2\sigma, \tag{14}$$
$$Mv = \kappa v - \lambda/\sigma, \tag{15}$$
$$Mw = \kappa w, \tag{16}$$

where $\kappa = (\rho_2 - \rho_0)g/\sigma$, and we can explicitly solve the equation for u, $\sin\psi = -\lambda r/4\sigma$ so that

$$\lambda = -4\sigma \sin\bar{\psi}/\bar{r} \tag{17}$$

The top surface is a sphere, and $v = V + \lambda/\kappa\sigma$ where V is as in the previous section. The equation (17) guarantees that the sphere meets the lower surface, and we write

$$F(\bar{\psi}) = v(\bar{r},\bar{\psi}) - w(\bar{r},\bar{\psi}) = V(\bar{r},\bar{\psi}) - \frac{4\sin\bar{\psi}}{\kappa\bar{r}} - w(\bar{r},\bar{\psi}). \tag{18}$$

Since $F(0) = -w(\bar{r},0) < 0$ and $F(\pi) = V(\bar{r},\pi) > 0$ we can deduce existence of a bubble as before. But we can do better. Using [10, theorem 7],

$$V(\bar{r},\frac{\pi}{2}) < \frac{1}{\kappa\bar{r}} + \left(\frac{2}{\kappa} + \frac{1}{(\kappa\bar{r})^2}\right)^{1/2} \tag{19}$$

and [10, theorem 14]

$$w(\bar{r},\frac{\pi}{2}) > -\frac{1}{\kappa\bar{r}} + \left(\frac{2}{\kappa} + \frac{1}{(\kappa\bar{r})^2}\right)^{1/2} \tag{20}$$

so $F(\pi/2) < -2/\kappa\bar{r}$ and there is a solution for $\bar{\psi} \in (\pi/2,\pi)$. This does not prohibit a sign change for F in $(0,\pi/2)$, but we have

Theorem 4.1 *$F(\bar{\psi}) < 0$ for $\bar{\psi} \in (0,\pi/2)$.*

Proof. From [10, theorem 7]

$$V(\bar{r},\bar{\psi}) < \frac{\sin\bar{\psi}}{\kappa\bar{r}} + \sqrt{\frac{2}{\kappa}(1-\cos\bar{\psi}) + \left(\frac{\sin\bar{\psi}}{\kappa\bar{r}}\right)^2} \tag{21}$$

and [10, theorem 14] together with the decrease of the exterior surfaces with respect to contact angle implies

$$w(\bar{r},\bar{\psi}) \geq T(\sigma) \geq -\frac{1}{\kappa\sigma} + \sqrt{\frac{2}{\kappa} + \left(\frac{1}{\kappa\sigma}\right)^2} \tag{22}$$

Since w bends over for $\bar{\psi} \in (0,\pi/2)$, [13, equation 2.9] implies

$$(r\sin\psi)_r = -\kappa r w \tag{23}$$

on the top, and integrating from σ to r,

$$\sigma - r\sin\psi = \kappa\int_\sigma^r \rho w d\rho > 0 \tag{24}$$

so that $\sigma > r\sin\psi$. Using $T(\sigma) \le w$ we can estimate the integral to obtain

$$\frac{\sigma - r\sin\psi}{\kappa} \ge T(\sigma)\frac{r^2-\sigma^2}{2} \tag{25}$$

ie.

$$r^2 - \sigma^2 \le 2\frac{\sigma - r\sin\psi}{\kappa T(\sigma)}. \tag{26}$$

We proceed by breaking up the interval into two parts. Consider first $[\pi/5, \pi/2)$. The above estimates imply

$$F(\bar\psi) < -\frac{3\sin\bar\psi}{\kappa\bar r} + \frac{1}{\kappa\sigma} + \sqrt{\frac{2}{\kappa}(1-\cos\bar\psi) + \left(\frac{\sin\bar\psi}{\kappa\bar r}\right)^2} - \sqrt{\frac{2}{\kappa} + \left(\frac{1}{\kappa\sigma}\right)^2}, \tag{27}$$

and $\sigma < r$ implies the difference of square roots is negative. The sum of the first two terms is negative if $\sigma \ge \bar r/3\sin\bar\psi$, and this is implied by $\bar r\sin\bar\psi \ge \bar r/3\sin\bar\psi$, ie. $\sin\bar\psi \ge 1/\sqrt{3}$, which is implied by $\bar\psi \ge \pi/5$.

For the interval $(0, \pi/5)$ we consider two cases depending on the magnitude of r. We have

$$F(\bar\psi) \le \sqrt{\frac{2}{\kappa}(1-\cos\bar\psi) + \left(\frac{\sin\bar\psi}{\kappa\bar r}\right)^2} - 3\frac{\sin\bar\psi}{\kappa\bar r}. \tag{28}$$

The right hand side is nonnegative if

$$g(\bar\psi) := \frac{4\sin^2\bar\psi}{\kappa\bar r^2} + \cos\bar\psi \ge 1. \tag{29}$$

Since $g(0) \ge 1$, this inequality holds as long as

$$g'(\bar\psi) = \frac{8}{\kappa\bar r^2}\sin\bar\psi\cos\bar\psi - \sin\bar\psi \ge 0 \tag{30}$$

ie. $\kappa\bar r^2/9 \le \cos\bar\psi$ which simplifies to $\bar r \le \sqrt{8\cos\bar\psi/\kappa}$.

This is our first case.

Since (17) holds on the top, equations (5) imply that

$$-(\cos\psi)_w + \frac{1}{r}\sin\psi = -w\kappa \tag{31}$$

there. For $\sigma \le \rho < r$, integrating from $w(\rho)$ to w yields

$$-\cos\psi(r) + \cos\psi(\rho) + \int_{w(\rho)}^{w(r)} \frac{\sin\psi}{r} du = -\kappa\frac{w^2(r) - w^2(\rho)}{2}, \tag{32}$$

and, letting $\rho \to \sigma$,

$$-\kappa[w^2(r) - T^2(\sigma)] = -2\cos\psi(r) + 2\int_{T(\sigma)}^{w} \frac{\sin\psi}{r} du. \tag{33}$$

Since $\sin\psi/r \leq 1/\sigma$

$$\kappa[w^2(r) - T^2(\sigma)] \geq 2\cos\psi(r) - 2\frac{w(r) - T(\sigma)}{\sigma}, \tag{34}$$

which implies

$$w \geq -\frac{1}{\sigma\kappa} + \sqrt{\frac{1}{\sigma^2\kappa^2} + \frac{2}{\kappa}\cos\psi + \frac{2}{\sigma\kappa}T + T^2}. \tag{35}$$

With this sharpened lower bound for w

$$F(\bar\psi) \leq -\frac{3\sin\bar\psi}{\kappa\bar r} + \sqrt{\frac{2}{\kappa}(1-\cos\bar\psi) + \frac{\sin^2\bar\psi}{\kappa^2\bar r^2}} + \frac{1}{\sigma\kappa} + \sqrt{\frac{1}{\sigma^2\kappa^2} + \frac{2}{\kappa}\cos\bar\psi + \frac{2}{\sigma\kappa}T + T^2}. \tag{36}$$

We estimate $F(\bar\psi)$ from above with the following steps. First replace the first square root with the sum of the square roots of the two terms to obtain $F(\bar\psi) \leq h(\bar\psi)$. We can rewrite $h(\bar\psi) < 0$ as

$$\sqrt{\frac{1}{\sigma^2\kappa^2} + \frac{2}{\kappa}\cos\bar\psi + \frac{2}{\sigma\kappa}T + T^2} > \frac{1}{\sigma\kappa} + \sqrt{\frac{2(1-\cos\bar\psi)}{\kappa}} - \frac{2\sin\bar\psi}{\kappa\bar r}. \tag{37}$$

Square both sides of this inequality and use again $\sin\bar\psi/\bar r \leq 1/\sigma$ to obtain

$$\frac{2}{\kappa}(2\cos\bar\psi - 1) + \frac{2}{\sigma\kappa}T + T^2 > \frac{2}{\sigma\kappa}\sqrt{\frac{2}{\kappa}(1-\cos\bar\psi)} - \frac{4\sin\bar\psi}{\kappa\bar r}\sqrt{\frac{2}{\kappa}(1-\cos\bar\psi)}. \tag{38}$$

On $(0, \pi/5)$ this is implied by

$$\frac{2}{\sigma\kappa}T + T^2 > \frac{2}{\sigma\kappa}\sqrt{\frac{2}{\kappa}((1-\cos\bar\psi)}. \tag{39}$$

The estimate (22) implies that

$$\frac{2}{\sigma\kappa}T + T^2 > \frac{2}{\kappa} \tag{40}$$

so that this last inequality is implied by

$$\sigma > \sqrt{\frac{2}{\kappa}(1-\cos\bar\psi)} \tag{41}$$

It remains to show that either (41) or $\bar{r} \leq \sqrt{8\cos\bar{\psi}/\kappa}$ holds. We have shown that $r^2 - \sigma^2 \leq 2(\sigma - r\sin\psi)/\kappa T$ on the top of the graph of w. Suppose that $r > \sqrt{8\cos\psi/\kappa}$ and $\sigma \leq \sqrt{2(1-\cos\psi)/\kappa}$. Then $r^2 - \sigma^2 > 2(5\cos\psi - 1)/\kappa$. To get a contradiction we need to show that

$$5\cos\psi - 1 \geq \frac{\sigma - r\sin\psi}{T}. \tag{42}$$

This is implied by

$$\frac{5\cos\psi - 1}{1 - \sin\psi} \geq \frac{\sigma}{T} \tag{43}$$

since $\sigma < r$. The left hand side of this inequality is greater than 4 on $(0, \pi/5)$, so we need only show that $\sigma/T \leq 4$. Using (20) this is implied by

$$\begin{aligned} \sigma^2 &\leq 4\left(-\frac{1}{\kappa} + \sqrt{\frac{2}{\kappa}\sigma^2 + \frac{1}{\kappa^2}}\right) \\ &= 4\frac{\frac{2}{\kappa}\sigma^2}{\frac{1}{\kappa} + \sqrt{\frac{2}{\kappa}\sigma^2 + \frac{1}{\kappa^2}}} \end{aligned}$$

which is, in turn, implied by $\sigma^2 \leq 24/\kappa$. It suffices to consider the largest value of σ under consideration, namely $\sigma = \sqrt{2(1-\cos\psi)/\kappa}$ and this inequality is satisfied for this choice of σ. The theorem is proven. ∎

We can also give an asymptotic result for smaller bubbles.

Theorem 4.2 *For $\epsilon > 0$ there is an $r_0 > 0$ such that $F(\bar{\psi}) < 0$ on $[\pi/2, \pi - \epsilon]$ for $r \leq r_0$.*

Proof. By lemma

$$F(\bar{\psi}) = -\frac{2\sin\bar{\psi}}{\kappa\bar{r}} + \mathcal{O}(\bar{r}) - w(\bar{r}, \bar{\psi}), \tag{44}$$

and [10, theorem 14] implies

$$\begin{aligned} F(\bar{\psi}) &\leq -\frac{\sin\bar{\psi}}{\kappa\bar{r}} + \mathcal{O}(\bar{r}) - \sqrt{\frac{2}{\kappa}(1 + \cos\bar{\psi}) + \frac{\sin^2\bar{\psi}}{\kappa\bar{r}}} \\ &\leq -\frac{\sin\bar{\psi}}{\kappa\bar{r}} + \mathcal{O}(\bar{r}). \end{aligned}$$

∎

Finally, we prove uniqueness of the solution for fixed $\bar{r}$. For this is suffices to show that

$$F(\bar{\psi}) = V - \frac{4\sin\bar{\psi}}{\kappa\bar{r}} - w \tag{45}$$

is increasing on $(\pi/2, \pi)$ since we have shown that $F < 0$ in $[0, \pi/2]$.

Theorem 4.3 *For fixed $\bar{r}$ there is exactly one bubble.*

Proof. We denote by $\Delta = \kappa r u - \sin\psi$ the denominator in equations (5). In finding the folded over surface V which is the bottom of the bubble u_0 must be varied so that $r(\bar{\psi}; u_0) = \bar{r}$. We denote the derivative with respect to u_0 by $(\dot{\ })$ as in [1, chapter 3]. Then, denoting derivatives with respect to $\bar{\psi}$ by $(\ )'$

$$\begin{aligned}\frac{dV}{d\bar{\psi}} &= V' + \dot{V} u_0' \\ &= V' + \dot{V}(\frac{-r'}{\dot{r}}) \\ &= \frac{r}{\dot{r}\Delta}(\dot{r}\sin\bar{\psi} - \dot{V}\cos\bar{\psi}) \\ &:= \frac{-r}{\dot{r}\Delta} f(\bar{\psi})\end{aligned}$$

where the implicit function theorem has been applied to $r(\bar{\psi}; u_0) - \bar{r} = 0$, and (5) has been used. The function $f(\psi)$ is zero exactly once in $(0, \pi)$ (always in $(\pi/2, \pi)$) and the solution ψ' is the contact point of the envelope of the family of solutions of (5) parameterized by u_0. For $\psi < \psi'$, $f(\psi) > 0$, so that for $\bar{\psi} < \psi'$, $dV/d\bar{\psi} > 0$, and $F(\bar{\psi})$ is increasing on $[\pi/2, \psi')$ since $w(\bar{r}, \bar{\psi})$ decreases as $\bar{\psi}$ increases from $\pi/2$.

It will suffice for us to show that $g(\bar{\psi}) := V(\bar{\psi}) - 4\sin\bar{\psi}/\kappa\bar{r}$ is increasing on (ψ', π).

Let us denote by $\mathcal{V}(\bar{\psi})$ the volume obtained by rotating V around the vertical axis. Then

$$\mathcal{V}(\bar{\psi}) = \pi\bar{r}(V\bar{r} - \frac{2}{\kappa}\sin\bar{\psi}). \tag{46}$$

We claim that $\mathcal{V}(\bar{\psi})$ is increasing in $\bar{\psi}$. Suppose for the moment that we assume this to be true. Then, for $\epsilon > 0$, $\mathcal{V}(\bar{\psi} + \epsilon) > \mathcal{V}(\bar{\psi})$ can be rewritten as

$$\mathcal{V}(\bar{\psi} + \epsilon) - \mathcal{V}(\bar{\psi}) > \frac{2}{\kappa\bar{r}}(\sin(\bar{\psi} + \epsilon) - \sin\bar{\psi}) \tag{47}$$

This implies

$$g(\bar{\psi} + \epsilon) - g(\bar{\psi}) > \frac{-2}{\kappa\bar{r}}(\sin(\bar{\psi} + \epsilon) - \sin\bar{\psi}) \tag{48}$$

and the right hand side is positive since $\bar{\psi} \in (\pi/2, \pi)$.

To see that $\mathcal{V}(\bar{\psi})$ increases for $\bar{\psi} \in (\pi/2, \pi]$ we need the fact that $\dot{\mathcal{V}}(\bar{\psi}) < 0$ which was proven by Finn [1, Theorem 3.2]. For fixed $\bar{r}$ increasing $\bar{\psi}$ decreases u_0 (since $\dot{r} < 0$, also proven by Finn). For a smaller value of u_0 and $\bar{r}$, $\bar{\psi}$ fixed the point on the graph of V is below and to the right and has bigger volume. Increasing $\bar{\psi}$ will make the volume even bigger. ■

Corollary 4.1 *There is a bubble of given volume V for each prescribed $V > 0$.*

Proof. We need only invoke estimates analogous to those in the proof of Theorem 3.2 to see that the volume goes to zero as $\bar{r}$ approaches zero and to infinity as $\bar{r}$ goes to infinity. ■

5 Open Problems

There are several questions which we have not been able to answer using our methods.

The simplest and perhaps most vexing one is uniqiness of drops of prescribed volume in the nonparameterized case ($\gamma_{02} < \pi/2$). Prescription of the volume is the natural physical condition, and it seems certain that this is true.

It would be desirable to show in general that the equation $F(\bar{\psi}; \bar{r}) = 0$ has a continuous branch of solutions for $0 < \bar{r} < \infty$. This then implies the existence of a drop with given volume. For bubbles the uniqueness in Theorem 4.3 makes this unnecessary and existence for a given volume is proven.

For small bubbles we have shown that the limiting case is a submerged bubble (Theorem 4.2). This was exhibited in numerical calculations [4]. In the opposite case of large bubbles the limiting case was a hemisphere standing on a flat surface. It would be interesting to prove this.

The uniqueness of a drop of given radius in the case $\pi/2 \leq \gamma_{02} \leq \pi$ seems plausable in view of the uniqueness for floating bubbles. In this case it may be useful to note that $F(\bar{\psi}; \bar{r})$ is increasing on the interval $[\gamma_{02} - \pi/2, \pi/2]$. However, computations show that we may have $F < 0$ at $\pi/2$.

References

[1] R. Finn, Equilibrium Capillary Surfaces, Springer-Verlag, New York, 1986.

[2] S.T. Gibbs, Ph.D. Thesis Research Proposal, University of Waterloo (1989).

[3] S. Hartland and R.W. Hartley, Axisymmetric Fluid-Liquid Interfaces, Elsevier, Amsterdam, 1976.

[4] Kevin Hemphill, Bubble Development of Known Radius from Capillary Surfaces, M.S. Thesis, Wichita State University, 1999.

[5] W.E. Johnson and L.M. Perko, Interior and exterior boundary value problems from the theory of the capillary tube, Arch. Rational Mech. Anal. **29** (1968), 125-143.

[6] A.D. Myshkis, V.G. Babskii, N.D. Kopachevskii, L.A. Slobozhanin and A.D. Typuptsov, Low-Gravity Fluid Mechanics, Springer-Verlag, New York, 1987.

[7] H.M.Princen, Shape of a fluid at a liquid-liquid interface, J. Colloid Science **18** (1963), 178-195.

[8] H.M.Princen and S.G. Mason, Shape of a fluid-liquid interface I. Extension and test of two-phase theory, J. Colliod Science **20** (1965), 156-172.

[9] H.M.Princen and S.G. Mason, Shape of a fluid-liquid interface II. Theory for three-phase systems, J. Colloid Science **20** (1965), 246-266.

[10] D. Siegel, Height estimates for capillary surfaces, Pacific J. Math. **88** (1980), 471-516.

[11] L.A. Slobozhanin, Equilibrium and stability of three capillary fluids with a common line of contact, Izvestiya Akademii Nauk SSSR, Mekhanika Zhidkosti i Gaza **176** no. 3 (1986), 170-173.

[12] B. Turkington, Height estimates for exterior problems of capillary type, Pacific J. Math. **88** (1980), 517-540.

[13] T.I. Vogel, Symmetric unbounded liquid bridges, Pacific J. Math. **103** (1982), 205-241.