Annular capillary surfaces

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Abstract

The meniscus in a symmetric annular capillary tube is investigated. The contact angles on the inner and outer tube surface need not be the same. Existence and qualitative properties of solutions are obtained using an iteration similar to that used by Johnson and Perko in the case of a circular capillary tube. If the contact angles have the same sign ideas of Siegel are used to give asymptotic estimates using circular arcs as comparison curves.

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1 Introduction

The equilibrium shapes of liquid-air interfaces have been the object of study for two hundred years. The mathematical description of these surfaces is given in terms of the Young-Laplace equation

$$div\left(\frac{grad\ u}{W}\right) = \kappa u$$

where $W = (1 + |grad\ u|^2)^{1/2}$ and $\kappa = \rho\sigma/g$ is the "capillary constant", where ρ is the density of the liquid, σ is the surface tension at the surface of the liquid, and g is the gravitational constant. The solutions of this equation and their properties have been of great interest to mathematicians, and general existence theory, state of the art numerical algorithms, and asymptotic methods have been applied. On the other hand, in contrast to the case of minimal surfaces, explicit solutions are not available even in the simplest geometries. In the fundamental case of a circular capillary tube knowledge is nearly complete. It is therefore surprising that a related special geometry, that of an "annular" tube, has not been extensively studied. There are some numerical and approximate results available ([4], [5], [6], [8]), but theoretical results known are mainly what can be derived from those for a general geometry. It is our purpose here to make a contribution to understanding this problem.

We consider here capillary surfaces bounded by two concentric circular cylindrical walls. The base domain is a circular annulus and we impose (possibly different) constant contact angles on each circle.

Known results imply existence and uniqueness of a solution of the implied nonlinear boundary value problem for the capillary surface equation ([1] [9]), but we study directly the ordinary differential equation boundary value problem for axisymmetric solutions. The methods that we use are closely related to those that have been used for circular cross section capillary tubes ([1] ch. 2, 3), and in the process of dealing with this problem directly we obtain a number of interesting properties of solutions. The existence proof uses techniques introduced by Johnson and Perko for the circular capillary tube. We first deal with a "half tube" problem in which the surface is horizontal at one end of its domain using [3], and then piece these together for the solution of the original problem. A simple computational algorithm is presented for solving the problem numerically which is based on the existence proof ideas. This has been implemented with a MATLAB code and numerical examples are presented.

In addition, we have obtained estimates for annular surfaces using circular arcs for comparison in the merideanal plane. Here we use the ideas of Siegel [7] in which the monotonicity of longitudinal curvature plays an essential role.

2 An existence theorem

The problem that we consider takes the form

$$\left(\frac{ru'}{\sqrt{1+(u')^2}}\right)' = r\kappa u, \quad 0 < a < r < b, \tag{1}$$

$$u'(a) = -\cot(\gamma_a), \quad u'(b) = \cot(\gamma_b), \tag{2}$$

where r is the radial coordinate, u is the surface height, and γ_a, γ_b are the contact angles at $r = a$, $r = b$, respectively, and $\kappa = \rho g/\sigma$, where ρ is the fluid density, g the gravitational constant, and σ the surface tension. We assume that $0 \leq \gamma_a, \gamma_b < \frac{\pi}{2}$ in what follows. The other combinations of possibilities will be dealt with later in the paper.

Without loss of generality we may introduce a scaling so that that $\kappa = 1$.

Since we have assumed $0 \leq \gamma_a, \gamma_b < \frac{\pi}{2}$ it is reasonable to assume that there is a unique m, $a < m < b$, at which $u'(m) = 0$. We take this as a starting point and consider the auxilliary problem

$$\left(\frac{ru'}{\sqrt{1+(u')^2}}\right)' = ru, \quad r > m, \tag{3}$$

$$u(m) = h, \quad u'(m) = 0, \tag{4}$$

where we assume $h > 0$. This problem is equivalent to the system

$$\begin{aligned} v(r) &= \frac{1}{r}\int_m^r u(\xi)\xi d\xi & (5)\\ u(r) &= h + \int_m^r \frac{v(\xi)}{\sqrt{1-v^2(\xi)}} d\xi & (6) \end{aligned}$$

which is a modification of the system considered in [3]. The solution of (3),(4) or (5),(6) can only be expected to exist on some interval, which will depend on h. Since u is increasing

$$v(r) < \frac{u(r)}{r}\int_m^r \xi d\xi < \frac{u(r)r}{2}$$

and

$$\frac{d}{dr}\left(\frac{v}{r}\right) = \frac{1}{r}\left(u - \frac{2v}{r}\right) > 0$$

for a solution of (5),(6).

We use the iteration

$$\begin{aligned} u_0 &= h, & (7)\\ v_n &= \frac{1}{r}\int_m^r u_n(\xi)\xi d\xi, & (8)\\ u_{n+1} &= h + \int_m^r \frac{v_n(\xi)}{\sqrt{1-v_n^2(\xi)}} d\xi. & (9) \end{aligned}$$

Then $v_0 = \frac{h}{2r}(r^2 - m^2)$ and $v_0(\rho_0) = 1$ where

$$\rho_0 = \frac{1 + \sqrt{1 + m^2h^2}}{h}.$$

The first iterant is then defined in $[m, \rho_0]$. It follows from (9) that $u_1 > u_0$, and u_1 is increasing on $[m, \rho_0]$. Also by (8), $v_1 > v_0$ there. It follows that there is a unique $\rho_1 < \rho_0$ where $v_1(\rho_1) = 1$ and $0 < v_1 < 1$ on (m, ρ_1). Since

$$\left(\frac{v_1}{r}\right)' = \frac{1}{r}\left(u_1 - \frac{2v_1}{r}\right)$$

and

$$v_1 < \frac{u_1 r}{2}$$

we see that v_1/r is increasing and $v_1 < r/\rho_1$ on $[m, \rho_1)$ so $v_1/\sqrt{1 - v_1^2}$ is integrable in $[m, \rho_1)$ and u_2 is finite at ρ_1.

Proceeding, we have $u_2 > u_1$, u_2 is increasing on $[m, \rho_1)$, $v_2 > v_1$ and there is a unique $\rho_2 < \rho_1$ such that $v_2(\rho_2) = 1$. Further $0 < v_2 < r/\rho_2$. The process continues and we have increasing sequences u_n, v_n, defined on $[m, \rho_n]$ where ρ_n is decreasing. Substitution in (9) of $v_n < r/\rho_n$ yields

$$u_{n+1} \leq h + \sqrt{\rho_n^2 - m^2} < h + \sqrt{\rho_0^2 - m^2} < h + \rho_0$$

so

$$v_{n+1} < \frac{h + \rho_0}{2r}(r_2 - m^2)$$

and

$$1 = v_{n+1}(\rho_{n+1}) < \frac{h + \rho_0}{2\rho_{n+1}}(\rho_{n+1}^2 - m^2)$$

which implies

$$\rho_{n+1} > (h + \rho_0)^{-1} + \sqrt{(h + \rho_0)^{-2} + m^2} > m.$$

This shows that $\rho = \lim \rho_n > m$ and we have $u = \lim u_n$, $v = \lim v_n$ defined on $[m, \rho]$ with $v(\rho) = 1$, $u(\rho)$ finite, but $u'(\rho) = \infty$.

We may think of the solution obtained of (3),(4) as a capillary surface with the prescribed height at m, contact angle $\frac{\pi}{2}$ there and contact angle 0 at some radius ρ larger than m. It is important for us to observe dependence of these quantities on h. We have, for $0 < h_1 < h_2$ that

$$\rho(h_1) \geq \rho(h_2), \tag{10}$$

$$\frac{u(r; h_1)}{h_1} \leq \frac{u(r; h_2)}{h_2}, \tag{11}$$

$$\frac{v(r; h_1)}{h_1} \leq \frac{v(r; h_2)}{h_2}. \tag{12}$$

These are shown by observing that $\rho_0(h)$ is decreasing and $u_0(r;h_1)/h_1 = u_0(r;h_2)/h_2$, $v_0(r;h_1)/h_1 = v_0(r;h_2)/h_2$, and then using (8), (9) inductively to see that (10), (11), (12) hold with ρ, u, v replaced by ρ_n, u_n, v_n. Then (10), (11), (12) follow by taking limits. We can also observe the important limits

$$\lim_{h\to\infty}\rho(h) = m, \quad \lim_{h\to 0}\rho(h) = \infty.$$

For $r < m$ we consider the analagous problem

$$\left(\frac{ru'}{\sqrt{1+(u')^2}}\right)' = ru, \quad r < m,$$

$$u(m) = h, \quad u'(m) = 0,$$

By almost exactly the same reasoning we obtain a solution on an interval (η, m) where $\eta = \eta(h)$, u is decreasing on (η, m) and $u'(\eta) = -\infty$. Further, $\lim_{h\to\infty}\eta(h) = m$, $\lim_{h\to 0}\eta(h) = -\infty$ and η is a decreasing function of h.

Our plan is to consider the joining of these two solutions on the interval (η, ρ) and to vary m and h to get solutions on a prescribed interval (a, b) with prescribed contact angles at the end points. (These solutions only make sense physically for values of h such that $\eta(h) > 0$.)

Theorem 2.1 *There is a solution of (1),(2).*

Proof. First we observe that for given m, there is a pair of numbers $a'(h) < m < b'(m)$ such that (2) are satisfied at a' and b' for $h > 0$. If $a'(h^*) = 0$, then we obtain annular surfaces for $h^*(m) < h < \infty$. We may assume that a', b' are differentiable as functions of h and that

$$\frac{da'}{dh} > 0, \quad \frac{db'}{dh} < 0. \tag{13}$$

We consider the set $A = \{0 \le a \le b\}$ in the ab plane. For each point on the line $a = b$ we may think of this coordinate as a (limiting) value of m. The values of a', b' as h varies from h^* to ∞ begin at some point on the b-axis. From (13) we may assume that $b' = b'(a')$ and that

$$\frac{db'}{da'} = \frac{\frac{db'}{dh}}{\frac{da'}{dh}} < 0.$$

We need then to see that these curves cover A simply. We need only observe that two distinct curves intersecting violates uniqueness for the capillary problem with those radii since the values of m would be distinct. ■

3 Computational examples

A MATLAB function was written in which values of m, h are given and the values of a, b at which (2) are satisfied are determined. This is particularly convenient if the curve $u = u(r)$ is parameterized with respect to the tangent angle ψ ([1], ch 3.):

$$\begin{aligned} \frac{du}{d\psi} &= \frac{r \sin\psi}{ru - \sin\psi}, \qquad (14) \\ \frac{dr}{d\psi} &= \frac{r \cos\psi}{ru - \sin\psi}, \qquad (15) \end{aligned}$$

$$r(0) = m, \quad u(0) = h. \qquad (16)$$

Since the contact angle is the complement of the tangent angle, we need only evaluate r at the appropriate values of ψ to obtain a and b. The initial value problems in (14)-(16) for $\psi > 0$ are solved using ODE45 from the MATLAB ODE suite. This determines a mapping from $m > 0$, $h > 0$ to values of a, b. To achieve a given pair $0 < a < b$ we used the MATLAB function LSQNONLIN from the optimization toolbox to determine the right values of m, h. Several cases are illustrated in figures 1, 2, and 3.

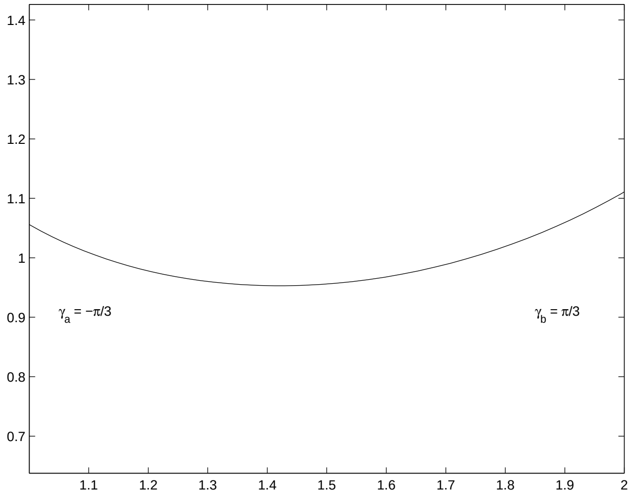

Figure 1: A surface in a small annulus.

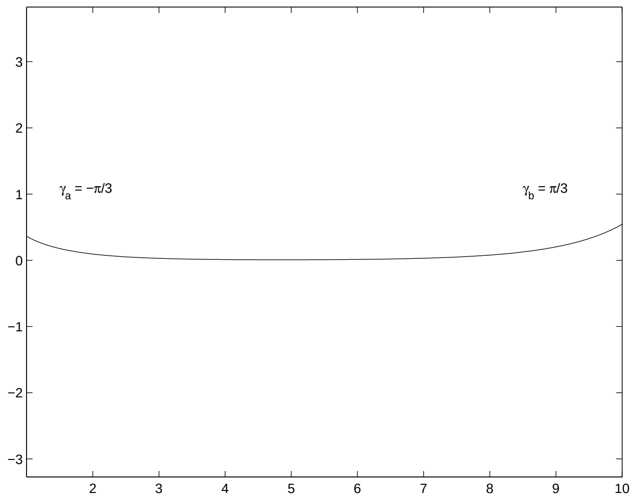

Figure 2: A surface in a large annulus.

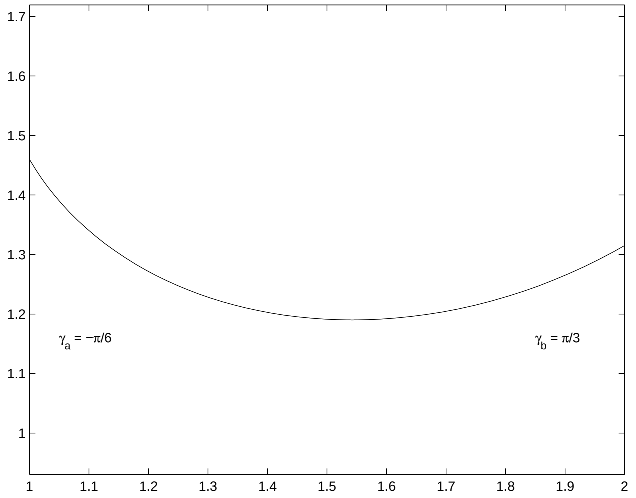

Figure 3: Differing values of contact angles.

4 Other configurations

If contact angles of opposite signs are given at the inner and outer radius then a somewhat different surface is obtained. For $0 < \gamma_a, \gamma_b < \frac{\pi}{2}$, we have

$$\left(\frac{ru'(r)}{\sqrt{1+u'(r)^2}}\right)' = \kappa r u \qquad a < r < b \tag{17}$$

$$u'(a) = \cot\gamma_a, \qquad u'(b) = \cot\gamma_b. \tag{18}$$

With (18), integrating (17) from a to r

$$\frac{ru'(r)}{\sqrt{1+u'(r)^2}} = \frac{a\cot\gamma_a}{\sqrt{1+\cot^2\gamma_a}} + \kappa\int_a^r u(\xi)\xi d\xi;$$

that is

$$\frac{u'(r)}{\sqrt{1+u'(r)^2}} = \frac{a}{r}\cos\gamma_a + \frac{\kappa}{r}\int_a^r u(\xi)\xi d\xi. \tag{19}$$

Denoting

$$v(r) = \frac{a}{r}\cos\gamma_a + \frac{\kappa}{r}\int_a^r u(\xi)\xi d\xi$$

and solving for $u'(r)$,

$$u'(r) = \frac{v(r)}{\sqrt{1-v(r)^2}}.$$

Integrating from a to r,

$$u(r) = u(a) + \int_a^r \frac{v(\xi)}{\sqrt{1-v(\xi)^2}} d\xi.$$

We consider the related initial value problem:

$$\left(\frac{ru'(r)}{\sqrt{1+u'(r)^2}}\right)' = r u(r) \qquad a < r, \tag{20}$$

$$u(a) = h, \qquad u'(a) = cot\gamma_a, \tag{21}$$

and the equivalent pair of integral equations

$$u(r) = h + \int_a^r \frac{v(\xi)}{\sqrt{1-v(\xi)^2}} d\xi, \tag{22}$$

$$v(r) = \frac{a}{r}\cos\gamma_a + \frac{\kappa}{r}\int_a^r u(\xi)\xi d\xi. \tag{23}$$

From (19), $u' > 0$, and the equations (22), (23) satisfy the initial conditions (21). Thus if $u(r)$ is the solution of (17), (18) on the interval $a < r < b$ then $u(r)$ and $v(r)$ satisfy (22), (23) and $|v(r)| < 1$ for $a < r < b$. Conversely, if $u(r)$ and

$v(r)$ are continuous functions and $|v(r)| < 1$ for $a < r < b$, then $u'(r)$ is continuous and $u(r)$ satisfies (17) on $a < r < b$.

If $|v(r)| \to 1$ as $r \to b$, then $|u'(r)| \to \infty$ as $r \to b$ and the solution of (20), (21) terminates at $r = b$. Fix the angle $\gamma_a < \frac{\pi}{2}$. If $h \geq 0$ then $u(b) > 0$ as $u' > 0$. If $h < 0$ and there exists m between a and b such that $u(m) = 0$ then $u(b) > 0$, because (22) implies

$$\begin{aligned} u(b) &= h + \int_a^b \frac{v(\xi)}{\sqrt{1 - v(\xi)^2}} d\xi \\ &= h + \int_a^m \frac{v(\xi)}{\sqrt{1 - v(\xi)^2}} d\xi + \int_m^b \frac{v(\xi)}{\sqrt{1 - v(\xi)^2}} d\xi \\ &= \int_m^b \frac{v(\xi)}{\sqrt{1 - v(\xi)^2}} d\xi \\ &> 0. \end{aligned}$$

If $h < 0$ and there does not exist m such that $u(m) = 0$, $a < m < b$, then $u(b) < 0$.

Theorem 4.1 *There exists a solution of (17), (18).*

Proof. We proceed with a series of lemmas.

Lemma 4.1 *For $h_\alpha < h_\beta$ we have $u(h_\alpha, b) < u(h_\beta, b)$ and $u'(h_\alpha, b) < u'(h_\beta, b)$.*

Proof. First we note that $u'(h_\alpha, a) = u'(h_\beta, a)$ as these are prescribed by the contact angle at a. Then (19) implies $u'(h_\alpha, r) < u'(h_\beta, r)$. We have $u'(h_\alpha, r) < u'(h_\beta, r)$ in a small interval $(a, a + \epsilon)$. But this shows that $u(h_\alpha, r) < u(h_\beta, r)$ in $(a, a + \epsilon)$ and thus we may extend the interval to $(a, b]$ by continuing the process. ■

Let $h_0 = sup\{h | u < 0 \text{ on } [a, b]\}$, the largest height for which the surface, if it exists, is below the axis. As we have seen, if $h < h_0$ then $u < 0$ on $[a, b]$.

Lemma 4.2 *There exists $c \in [a, b]$ such that $u(h_0, c) = 0$ and $u'(c) > 0$, then $u' > 0$ for $r > c$.*

Proof. There must be a $c \in [a, b]$ such that $u(c) = 0$. If this were not true, we would have the whole surface below 0, and h_0 would not be the supremum. For $u'(c) > 0$ to hold we observe that to the left of c, u must be increasing. But we know that $u \equiv 0$ is not a solution, so we may rule out the case that $u'(c) = 0$, leaving $u'(c) > 0$. Integrating (17) from c to r:

$$\frac{ru'}{\sqrt{1 + (u')^2}} = \frac{cu'(c)}{\sqrt{1 + (u'(c))^2}} + \int_c^r \kappa \xi u(\xi) d\xi.$$

Thus $u'(r) > 0$ on $[c, b]$. ■

Lemma 4.3 *There exist h_1 and h_2 such that $h_1 \leq h \leq h_2$ implies that there exist $u(h,b), u'(h,b) < \infty$.*

Proof. If we take h_1 from inverting Theorem 2.1 with some negative slope at b, we have a surface that extends to b. Now from continuous dependence on the data, h, we may find an $h_2 > h_1$ as follows. For δ sufficiently small we have $u'(h_1 + \delta, b) < u'(h_1, b) + \epsilon$ and $u(h_1 + \delta, b) < u(h_1, b) + \epsilon$ for a given $\epsilon > 0$. Now we need only pick ϵ small enough that the right hand side of both of these inequalities are finite. We then set $h_2 = h_1 + \delta$. By Lemma 4.1, we may assert the claim for any h in $[h_1, h_2]$. ■

We define $\tilde{h} = \sup h_2$.

Lemma 4.4 $u'(b, \tilde{h}) = \infty$

Proof. By Lemma 4.1, as we increase h_2 we increase $u'(h_2, b)$. Thus either $u'(\tilde{h}, b) = \infty$ or $u'(\tilde{h}, b) < \infty$. To show this limit must be unbounded we take $u'(\tilde{h}, b) < \infty$ and then take $\tilde{h} + \epsilon$ with $\epsilon > 0$ small enough and see that $u'(\tilde{h} + \epsilon, b) < \infty$, and $u(\tilde{h}, b) < \infty$, thus we have a contradiction as $\tilde{h}$ was not the supremum. ■

We have shown that $\{u'(h,b) | h \leq \tilde{h}\} \supset [0, \infty)$, hence a solution of (17), (18) exists for any $0 < \gamma_b < \frac{\pi}{2}$. ■

The above process is illustrated by the computations shown in figure 4.

All other cases for $\gamma_a, \gamma_b \in [0, \pi]$ can be derived from the two cases already considered by inversion or from the case in which the surface is horizontal at an end point which reduces to our auxilliary problem.

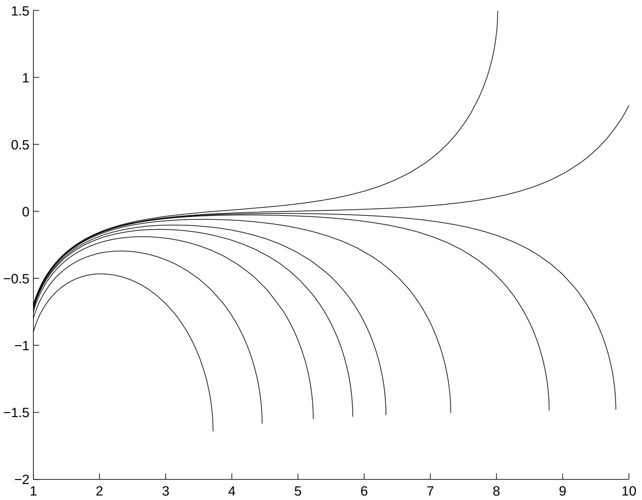

Figure 4: u and u' increasing as h increases, sweeping out all the angles at the radius b.

5 Estimates for $0 < \gamma_a < \frac{\pi}{2}$, $0 < \gamma_b < \frac{\pi}{2}$

As stated in the introduction we now give estimates that can be derived using circular arcs for comparison.

It is useful to introduce dimensionless variables at this point. Since we are considering surfaces of the kind studied in Section 2 we know that there is a unique m, $a < m < b$ such that $u'(m) = 0$. We choose m as the reference length for scaling and (1), (2) then become

$$(r \sin \psi)_r = Bru \quad \delta < r < \delta', \tag{24}$$

$$\sin(\psi(\delta)) = -\cos \gamma_a, \quad \sin(\psi(\delta')) = \cos \gamma_a \tag{25}$$

where $\delta < 1 < \delta'$, $B = m^2 \kappa$, and u, r have been scaled in the usual way. There are two dimensionless parameters for this problem which we take to be B and δ. The right endpoint δ' is determined by B and δ in this formulation. (Other normalizations might seem more natural at first look, but we have found the results more clear using this one.)

We give seperate estimates on $(\delta, 1)$ and $(1, \delta')$, in that order.

If we integrate (24), (25) and use the fact that u decreases on $(\delta, 1)$ the estimates

$$B\frac{u_1}{2}\left(\frac{1}{r^2} - 1\right) < -\frac{\sin \psi}{r} < B\frac{u}{2}\left(\frac{1}{r^2} - 1\right) \tag{26}$$

follow (cf. [1] ch.3.) for $\delta < r < 1$. The equation (24) can be written

$$\frac{\sin \psi}{r} + (\sin \psi)_r = k_l + k_m = Bu \tag{27}$$

where k_l, k_m are the longitudinal and merideanal curvatures. We can derive

$$(k_l)_r = \frac{2}{r}\left(B\frac{u}{2} - \frac{\sin \psi}{r}\right) > 0 \tag{28}$$

using (26). In the case of a circular capillary tube $(k_m)_r > 0$ as well and Finn used this in deriving a number of estimates. Unfortunately, this does not appear to be true here, but we have been able to adapt the techniques of Siegel [7] to the present situation since he used only $(k_l)_r > 0$ in giving estimates for the circular capillary tube.

We use circular arcs which are horizontal at 1 an satisfy $\sin(\psi(\delta)) = -\cos \gamma_a$. If we denote $u(1) = u_1$ and $u(\delta) = u_\delta$, these are

$$S_1 = -\sqrt{(1-\delta)^2 \sec^2 \gamma_a - (r-1)^2} + (1-\delta) \tan \gamma_a + u_\delta \tag{29}$$

and

$$S_2 = -\sqrt{(1-\delta)^2 \sec^2 \gamma_a - (r-1)^2} + (1-\delta) \sec \gamma_a + u_1 \tag{30}$$

It follows as in [7] that (28) implies

$$S_1 < u < S_2 \tag{31}$$

since $(k_l)_r > 0$ and the curvatures of S_1, S_2 are constant, $S_1(\delta) = u_\delta$, $S_2(1) = u_1$.

If we multiply (31) by r and integrate from δ to 1, we obtain

$$\int_\delta^1 rS_1(r)dr < \frac{\cos(\gamma_a)}{B}\delta < \int_\delta^1 rS_2(r)dr. \tag{32}$$

The expressions on the left and right can be integrated explicitly yielding

$$\begin{aligned}\int_\delta^1 rS_1 dr &= -\frac{(1-\delta)^2\sec^2\gamma_a}{2}(\frac{\pi}{2}-\gamma_a) - \frac{(1-\delta)^2\tan\gamma_a}{2} + \\ &\quad \frac{(1-\delta^2)}{2}(u_\delta + (1-\delta)\tan\gamma_a) + \frac{(1-\delta)^3}{3}\frac{1-\sin^3\gamma_a}{\cos^3\gamma_a},\end{aligned} \tag{33}$$

$$\begin{aligned}\int_\delta^1 rS_2 dr &= -\frac{(1-\delta)^2\sec^2\gamma_a}{2}(\frac{\pi}{2}-\gamma_a) - \frac{(1-\delta)^2\tan\gamma_a}{2} + \\ &\quad \frac{(1-\delta^2)}{2}(u_1 + (1-\delta)\sec\gamma_a) + \frac{(1-\delta)^3}{3}\frac{1-\sin^3\gamma_a}{\cos^3\gamma_a}.\end{aligned} \tag{34}$$

If the two inequalities are solved, respectively, for u_1, u_δ we obtain

$$\begin{aligned}u_1 &> \frac{2\delta\cos\gamma_a}{B(1-\delta^2)} + \frac{(1-\delta)\sec^2\gamma_a}{1+\delta}(\frac{\pi}{2}-\gamma_a) + \frac{(1-\delta)\tan\gamma_a}{1+\delta} \\ &\quad -\frac{2}{3}\frac{(1-\delta)^2}{1+\delta}\frac{1-\sin^3\gamma_a}{\cos^3\gamma_a} - (1-\delta)\sec\gamma_a \\ &:= \mathcal{L}_1\end{aligned} \tag{35}$$

$$\begin{aligned}u_\delta &< \frac{2\delta\cos\gamma_a}{B(1-\delta^2)} + \frac{(1-\delta)\sec^2\gamma_a}{1+\delta}(\frac{\pi}{2}-\gamma_a) + \frac{(1-\delta)\tan\gamma_a}{1+\delta} \\ &\quad -\frac{2}{3}\frac{(1-\delta)^2}{1+\delta}\frac{1-\sin^3\gamma_a}{\cos^3\gamma_a} - (1-\delta)\tan\gamma_a \\ &:= \mathcal{L}_2\end{aligned} \tag{36}$$

The first of these should be compared with Laplace's formula for a circular tube ([1] ch. 2.):

$$\mathcal{L} := \frac{2\cos\gamma}{B} - \frac{1}{\cos\gamma} + \frac{2}{3}\frac{1-\sin^3\gamma}{\cos^3\gamma}. \tag{37}$$

(The difference in sign is due to our consideration of the left side.)

We introduce another circular arc

$$\begin{aligned}S(r) &= -(1-\delta)\sqrt{\sec^2\gamma_a - \left(\frac{r-1}{\delta-1}\right)^2} + \frac{2\delta\cos\gamma_a}{B(1-\delta^2)} + \frac{(1-\delta)\sec^2\gamma_a}{1+\delta}(\frac{\pi}{2}-\gamma_a) \\ &\quad + \frac{(1-\delta)\tan\gamma_a}{1+\delta} - \frac{2}{3}\frac{(1-\delta)^2}{1+\delta}\frac{1-\sin^3\gamma_a}{\cos^3\gamma_a}\end{aligned} \tag{38}$$

This one satisfies $S(1) = \mathcal{L}_1$, $S(\delta) = \mathcal{L}_2$ and

$$|u - S| \le \max\{u_1 - \mathcal{L}_1, \mathcal{L}_2 - u_\delta\}. \tag{39}$$

From (35), (36) we obtain

$$\begin{aligned} u_\delta - u_1 < \mathcal{L}_2 - \mathcal{L}_1 &= (1-\delta)(\sec\gamma_a - \tan\gamma_a) \\ &= (1-\delta)\frac{1-\sin\gamma_a}{\cos\gamma_a}. \end{aligned} \tag{40}$$

(cf. [7] for a similar estimate.) As $u_r < 0$ for $\delta < r < 1$, we may use u as the independent variable. We have $(\sin\psi)_r = -(\cos\psi)_u$ from the chain rule. Equation (27) can now be written as

$$-(\cos\psi)_u + \frac{\sin\psi}{r} = Bu. \tag{41}$$

Using the first part of (26) we have

$$\int_{u_\delta}^{u_1} \frac{\sin\psi}{r} < \int_{u_\delta}^{u_1} \frac{Bu_1}{2}(1-\frac{1}{r^2}) = \frac{Bu_1}{2}(1-\frac{1}{\delta^2})(u_1-u_\delta) \tag{42}$$

Integrating (41), we obtain from (42)

$$\sin\gamma_a - 1 + \frac{Bu_1}{2}(1-\frac{1}{\delta^2})(u_1-u_\delta) > \frac{B}{2}(u_1^2-u_\delta^2)$$

Thus (35), (36) imply

$$2\frac{1-\sin\gamma_a}{B(\mathcal{L}_2+\frac{1}{\delta^2}\mathcal{L}_1)} < u_\delta - u_1. \tag{43}$$

Now

$$\begin{aligned} u_1 - \mathcal{L}_1 &= u_\delta - \mathcal{L}_1 - (u_\delta - u_1) \\ &< \mathcal{L}_2 - \mathcal{L}_1 - 2\frac{1-\sin\gamma_a}{B(\mathcal{L}_2+\frac{1}{\delta^2}\mathcal{L}_1)} \end{aligned} \tag{44}$$

and

$$\begin{aligned} \mathcal{L}_2 - u_\delta &= \mathcal{L}_2 - u_1 - (u_\delta - u_1) \\ &< \mathcal{L}_2 - \mathcal{L}_1 - 2\frac{1-\sin\gamma_a}{B(\mathcal{L}_2+\frac{1}{\delta^2}\mathcal{L}_1)} \end{aligned} \tag{45}$$

imply that

$$\begin{aligned} |u(r) - S^-(r)| &\le \mathcal{L}_2 - \mathcal{L}_1 - 2\frac{1-\sin\gamma_a}{B(\mathcal{L}_2+\frac{1}{\delta^2}\mathcal{L}_1)} \\ &= (1-\delta)(\tan\gamma_a - \sec\gamma_a) + \frac{(\sin\gamma_a - 1)\delta(1-\delta)}{(1+\delta^2)\cos\gamma_a}\frac{1}{1-f(\delta,\gamma_a)B} \end{aligned} \tag{46}$$

where

$$\begin{aligned} f(\delta,\gamma_a) \quad &:= \quad \frac{-(\delta-1)^2\sec^2\gamma_a}{2\delta\cos\gamma_a}(\frac{\pi}{2}-\gamma_a)+\frac{(\delta-1)(1-\delta^2)\tan\gamma_a}{2\delta\cos\gamma_a} \\ &\quad +\frac{2}{3}\frac{(1-\delta)^3}{2\delta}\frac{1-\sin^3\gamma_a}{\cos^3\gamma_a}+\frac{(\delta-1)(1-\delta^2)(\delta^2\sin\gamma_a+1)}{2\delta(1+\delta^2)\cos\gamma_a}. \end{aligned} \tag{47}$$

Since f does not depend on B we take the limit as $B \to 0$ on the right hand side, obtaining

$$(1-\delta)(\tan\gamma_a-\sec\gamma_a)+\frac{(1-\sin\gamma_a)\delta(1-\delta^2)}{2(1+\delta^2)\cos\gamma_a}.$$

Thus for $B \to 0$ our estimate does not imply that u is approaching a circular arc, in contrast to the case of a circular capillary tube.

Looking at (46), this time considering $\delta \to 1$. First we see that $f(\delta,\gamma^-) \to 0$ as $\delta \to 1$. Thus $u(r) \to S(r)$ as $\delta \to 1$.

Next we look at the right part of the domain. We find very similar results to the left side, but we need to first look into the limiting behavior of δ'.

Lemma 5.1 *If* $\delta \to 1$ *for fixed* B*, so does* δ'*.*

Proof. From (35) we see that as $\delta \to 1, u_1 \to \infty$. Integrating (24) from 1 to $r \le \delta'$ as $u'(1) = 0$ we have

$$\frac{ru'}{\sqrt{1+(u')^2}}=\int_1^r B\eta u(\eta)d\eta. \tag{48}$$

So that $\delta'\cos\gamma_b \ge B\frac{u_1}{2}((\delta')^2-1/\delta')$. Since $u_1 \to \infty$ we must have $\delta' \to 1$. ■

We define for $r \in (1,\delta')$

$$S_1'(r)=(\delta'-1)\left(-\sqrt{\sec^2\gamma_b-\left(\frac{r-1}{\delta'-1}\right)^2}+\tan\gamma_b\right)+u_{\delta'} \tag{49}$$

and

$$S_2'(r)=(\delta'-1)\left(-\sqrt{\sec^2\gamma_b-\left(\frac{r-1}{\delta'-1}\right)^2}+\sec\gamma_b\right)+u_1. \tag{50}$$

We have again

$$S_1'(r)<u(r)<S_2'(r) \quad \text{for } 1<r<\delta'. \tag{51}$$

Now multiplying this by r, integrating from 1 to δ', and solving for u_1 and $u_{\delta'}$ respectively, as $\delta' > 1$ we have

$$\begin{aligned} u_1 \quad &> \quad -(\delta'-1)\sec\gamma_b+\frac{2\delta'\cos\gamma_b}{B((\delta')^2-1)}+\frac{(\delta'-1)\sec^2\gamma_b}{1+\delta'}(\frac{\pi}{2}-\gamma_b) \\ &\quad +\frac{(\delta'-1)\tan\gamma_b}{1+\delta'}+\frac{2}{3}\frac{(\delta'-1)^2}{1+\delta'}\frac{1-\sin^3\gamma_b}{\cos^3\gamma_b} \end{aligned}$$

$$\begin{aligned}
&:= \mathcal{L}_1' && (52)\\
u_{\delta'} &< -(\delta'-1)\tan\gamma_b + \frac{2\delta'\cos\gamma_b}{B((\delta')^2-1)} + \frac{(\delta'-1)\sec^2\gamma_b}{1+\delta'}(\frac{\pi}{2}-\gamma_b)\\
&\quad + \frac{(\delta'-1)\tan\gamma_b}{1+\delta'} + \frac{2}{3}\frac{(\delta'-1)^2}{1+\delta'}\frac{1-\sin^3\gamma_b}{\cos^3\gamma_b}\\
&:= \mathcal{L}_2' && (53)
\end{aligned}$$

Here compare the first with Laplace's formula: (37).

The analogue to (38) is

$$\begin{aligned}
S'(r) &= -(\delta'-1)\sqrt{\sec^2\gamma_b - \left(\frac{r-1}{\delta'-1}\right)^2} + \frac{2\delta'\cos\gamma_b}{B((\delta')^2-1)} + \frac{(\delta'-1)\sec^2\gamma_b}{1+\delta'}(\frac{\pi}{2}-\gamma_b)\\
&\quad + \frac{(\delta'-1)\tan\gamma_b}{1+\delta'} + \frac{2}{3}\frac{(\delta'-1)^2}{1+\delta'}\frac{1-\sin^3\gamma_b}{\cos^3\gamma_b}, && (54)
\end{aligned}$$

where $S'(1) = \mathcal{L}_1'$ and $S'(\delta') = \mathcal{L}_2'$.

We have

$$|u(r) - S'(r)| \le \max\{u_1 - \mathcal{L}_1', \mathcal{L}_2' - u_\delta\} \qquad (55)$$

and as before

$$\begin{aligned}
u_{\delta'} - u_1 &< S'(\delta') - S'(1)\\
&= (\delta'-1)(\frac{1-\sin\gamma_b}{\cos\gamma_b}). && (56)
\end{aligned}$$

As $u_r > 0$ for $1 < r < \delta'$, we may use u as the independent variable again. So (41) is still valid in the right part of the domain. Now integrating (41) and using the

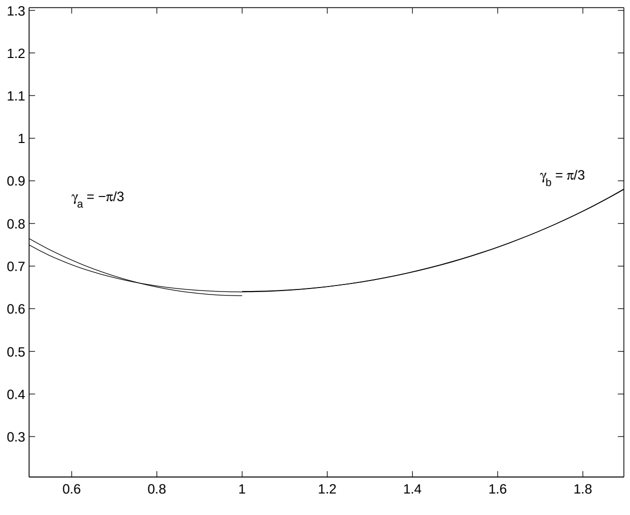

Figure 5: The comparison arcs S and S' with an annular surface with $B = 1$. Note the right arc is nearly indistinguishable from the surface.

first part of (26) we obtain

$$2\frac{1-\sin\gamma_b}{B(\mathcal{L}_2' + \frac{1}{(\delta')^2}\mathcal{L}_1')} < u_{\delta'} - u_1 \tag{57}$$

in exactly the same way as before.

Now

$$\begin{aligned} u_1 - \mathcal{L}_1' &= u_{\delta'} - \mathcal{L}_1' - (u_{\delta'} - u_1) \\ &< \mathcal{L}_2' - \mathcal{L}_1' - 2\frac{1-\sin\gamma_b}{B(\mathcal{L}_2' + \frac{1}{(\delta')^2}\mathcal{L}_1')} \end{aligned} \tag{58}$$

and

$$\begin{aligned} \mathcal{L}_2' - u_{\delta'} &= \mathcal{L}_2' - u_1 - (u_{\delta'} - u_1) \\ &< \mathcal{L}_2' - \mathcal{L}_1' - 2\frac{1-\sin\gamma_b}{B(\mathcal{L}_2' + \frac{1}{(\delta')^2}\mathcal{L}_1')} \end{aligned} \tag{59}$$

as before (46) implies that

$$\begin{aligned} |u(r) - S'(r)| &\leq \mathcal{L}_2' - \mathcal{L}_1' - 2\frac{1-\sin\gamma_b}{B(\mathcal{L}_2' + \frac{1}{(\delta')^2}\mathcal{L}_1')} \\ &= (1-\delta')(\tan\gamma_b - \sec\gamma_b) \\ &\quad + \frac{(\sin\gamma_b - 1)\delta'(1-\delta')}{(1+(\delta')^2)\cos\gamma_b}\frac{1}{1 - f(\delta',\gamma_b)B} \end{aligned} \tag{60}$$

where f is defined by (47) with δ and γ_a replaced with δ' and γ_b respectively.

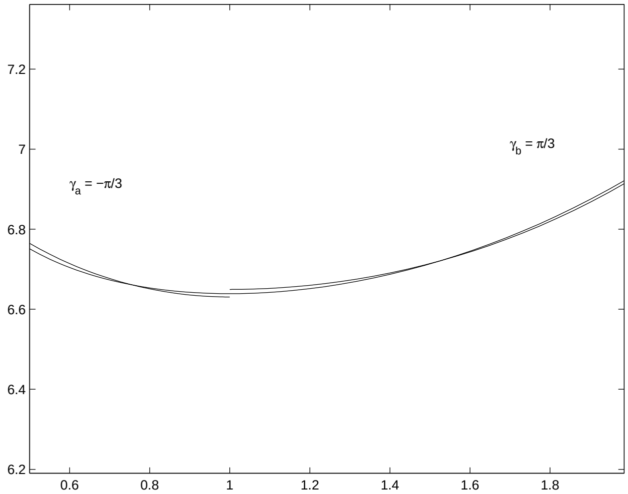

Figure 6: The comparison arcs S and S' with an annular surface with $B = 0.1$.

Again $B \to 0$ does not imply that $u(r) \to S'(r)$. However $u(r) \to S'(r)$ as $\delta' \to 1$ for fixed B, and we have seen $\delta' \to 1$ as $\delta \to 1$.

In figures 5 through 8 there are several examples of our estimation with two sets of contact angles and two values of B.

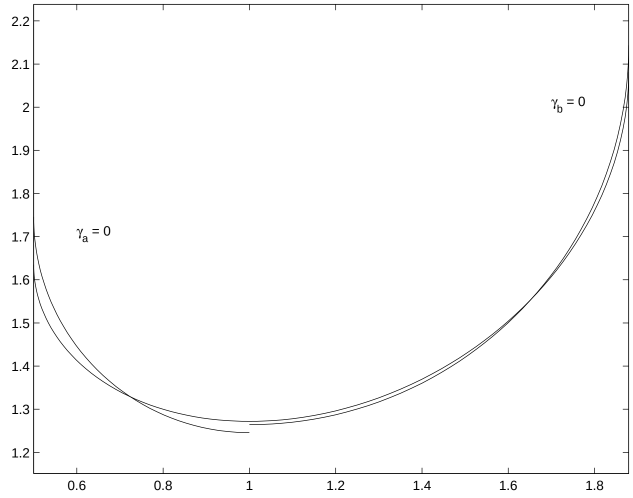

Figure 7: The comparison arcs S and S' with an annular surface with $B = 1$ and a different case of contact angles.

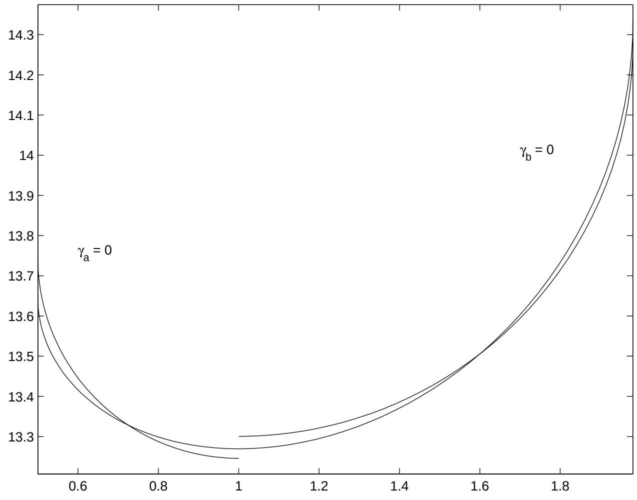

Figure 8: The comparison arcs S and S' with an annular surface with $B = 0.1$ and the different case of contact angles.

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