

Some Notes on Area

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The problem: Given the graph of a continuous function $y = f(x)$, with $f(x) > 0$ for all x in the closed interval $[a, b]$, find the area of the region bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = a$, and $x = b$.

In other words, find the area under the curve $y = f(x)$ from a to b .

Seriously, it's a problem: Mathematicians had been trying to find an answer to the above problem since long before the discovery (or invention – depending on your philosophy) of calculus. It is very hard to find the area of shapes with curved edges. As an example, try to find the area of a disc with radius 1. A circle is a very simple shape, so you wouldn't think that this would be a difficult problem. However it turned out to be an extremely difficult and perplexing problem for ancient mathematicians. In fact, it is one of the problems that eventually led to the discovery (or invention...) of the number π . The actual problem that gave us the number π was to find the circumference a circle. In fact, π is *defined* to be the circumference divided by diameter of any circle; $\pi = \frac{c}{d}$. Today, when we are asked to find the area of the unit circle we just say π . We don't even have to think very hard. But we usually don't think about the fact that we can't even write down the number π .

A π interlude: A good math student never (ab)uses a calculator. We all know that π is approximately equal to 3.14159..., and that no matter how many decimal places we carry this out, we will never actually get π . So if we need to approximate an answer we usually just write $\pi \approx 3.14$.

Here is a better approximation. Write down the number 113355, and then break it down the middle: 113 | 355. Now make it into a fraction by "twisting" it counter-clockwise: $\frac{355}{113}$. This fraction agrees with π through the first 5 decimal places! So it's a much better approximation than 3.14. Plus it's a fraction, and fractions are always better than decimals. It's also easy to remember, and just plain awesome.

Let's get back to the problem at hand. We were trying to find the area underneath the curve $y = f(x)$.

A great idea: We can approximate the area under the curve by *partitioning* the interval $[a, b]$ into many segments, then drawing rectangles above each segment that are approximately the same height as the function. Rectangles are easy to find the area of, so we find the area of each one and then add them all up.

Suppose we partition the interval into n segments with endpoints $a = x_0, x_1, \dots, x_n = b$ as in the picture above. We can then choose any point x_i^* in the interval $[x_{i-1}, x_i]$, and make the height of the rectangle $f(x_i^*)$ on that segment. Let $\Delta x_i = x_i - x_{i-1}$ denote the length of the segment. The area of a rectangle equals base times height, so the area of the rectangle above the i^{th} segment is given by $f(x_i^*) \cdot \Delta x_i$. We have n segments, so to approximate the area under the curve we have to find the area of all n rectangles and then add them up. We get

$$\text{Area} \approx f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \dots + f(x_n^*)\Delta x_n,$$

or written in Sigma notation

$$\text{Area} \approx \sum_{i=1}^n f(x_i^*)\Delta x_i.$$

This is called a (finite) *Riemann sum*, named after the famous German mathematician Bernhard Riemann. It really is a great idea, and it's always possible to calculate a finite Riemann sum explicitly. It can be a real pain though. To make things easier, we usually use a uniform partition. That is, when we break up the interval $[a, b]$ into n pieces we make all of the pieces the same length, $\Delta x = \frac{b-a}{n}$. We also usually take the x_i^* to be either right endpoints, left endpoints, or midpoints of the partition segments. We get the following area approximation formulas

$$\begin{aligned} R_n &= \sum_{i=1}^n f(x_i) \frac{b-a}{n}, \\ L_n &= \sum_{i=1}^n f(x_{i-1}) \frac{b-a}{n}, \text{ and} \\ M_n &= \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) \frac{b-a}{n}. \end{aligned}$$

Example: Let $f(x) = x^2$, $a = 0$, and $b = 2$. Find R_4 , L_4 , and M_4 .

First we need to partition the interval $[0, 2]$ into four equal segments. The length of each segment is given by $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$. The initial partition point is $x_0 = a = 0$. The i^{th} partition point is given by the formula

$$x_i = a + i\Delta x.$$

In this case we have $x_i = 0 + \frac{1}{2}i = \frac{1}{2}i$, for $i = 1, 2, 3, 4$. So we have partitioned the interval $[0, 2]$ into four equal parts with partition points

$$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1, \quad x_3 = \frac{3}{2}, \quad x_4 = 2,$$

and subintervals

$$\left[0, \frac{1}{2}\right], \quad \left[\frac{1}{2}, 1\right], \quad \left[1, \frac{3}{2}\right], \quad \left[\frac{3}{2}, 2\right].$$

The rest of the problem is just *plug and chug*. We have all of the ingredients, we just have to compute the answers. We get

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) \\ &= \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}(1)^2 + \frac{1}{2}\left(\frac{3}{2}\right)^2 + \frac{1}{2}(2)^2 \\ &= \frac{1}{2}\left(\frac{1}{4} + 1 + \frac{9}{4} + 4\right) \\ &= \frac{1}{2}\left(\frac{34}{4}\right) = \frac{17}{4}, \end{aligned}$$

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) \\ &= \frac{1}{2}(0)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}(1)^2 + \frac{1}{2}\left(\frac{3}{2}\right)^2 \\ &= \frac{1}{2}\left(0 + \frac{1}{4} + 1 + \frac{9}{4}\right) \\ &= \frac{1}{2}\left(\frac{14}{4}\right) = \frac{7}{4}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} M_4 &= f\left(\frac{x_0+x_1}{2}\right)\Delta x + f\left(\frac{x_1+x_2}{2}\right)\Delta x + f\left(\frac{x_2+x_3}{2}\right)\Delta x + f\left(\frac{x_3+x_4}{2}\right)\Delta x \\ &= \frac{1}{2}f\left(\frac{1}{4}\right) + \frac{1}{2}f\left(\frac{3}{4}\right) + \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f\left(\frac{7}{4}\right) \\ &= \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{5}{4}\right)^2 + \frac{1}{2}\left(\frac{7}{4}\right)^2 \\ &= \frac{1}{2}\left(\frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16}\right) \\ &= \frac{1}{2}\left(\frac{84}{16}\right) = \frac{21}{8}. \end{aligned}$$

□

Exercise: Draw pictures of each of the approximations R_4 , L_4 , and M_4 . Notice that R_4 is an over approximation and L_4 is an under approximation. M_4 is somewhere in between, but it's not clear whether it is an over or under approximation. We will see later that the actual area under this curve is $\frac{8}{3}$, so M_4 turns out to be an under estimate. But look at how close of an approximation it is. It's only off by $\frac{1}{24}$!

Exercise: Find R_{10} , L_{10} , and M_{10} for $y = x^2$, $a = 0$, and $b = 2$. Compare your results to the previous example. What do you find?

Smaller is better: We can get a better approximation of the area by taking more partition points, so that the resulting subintervals are smaller. Then the rectangles will be more narrow, and the error will be smaller. The catch is that we will have more rectangles to calculate the area of and then add up.

Call in the limits: We need help. We want to use this method of approximating area by rectangles to find the exact area under the curve. But as long as we are using a finite number of rectangles, then we will always have some error (albeit small) in our calculation. We've just seen that more boxes are better. The idea is to find a general formula for n rectangles, and then take the limit as n goes to infinity. Thus the *exact* area under the curve is given by the *Riemann sum*

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Since we are taking the limit as n tends to infinity, it doesn't matter how we choose the x_i^* ; we'll get the same answer no matter how we choose them. In particular,

$$\text{Area} = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} M_n.$$

From now on we will use a uniform partition with right endpoints, R_n , but it is important to remember that you could use any other choice of x_i^* . Our formula for area under a curve is then

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x.$$

Remembering our previous work, we have the following formulas

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad x_i = a + i\Delta x = a + \frac{b-a}{n} \cdot i.$$

Plugging these into the area formula, we get

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(a + \frac{b-a}{n} \cdot i\right) \frac{b-a}{n}.$$

Since the sum is over i , and $\frac{b-a}{n}$ does not depend on i , we can pull it out of the sum. Then our formula for area under a curve becomes

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n f\left(a + \frac{b-a}{n} \cdot i\right)$$

Some Useful Formulæ: We will need the following summation formulas if we are actually going to calculate the exact area under a curve:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

There are many more, but these will do for now.

Exercise: Prove the first formula.

Example: Find the exact area under the curve $f(x) = x^2$ between $x = 0$ and $x = 2$.

First we need to find Δx and x_i ,

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}, \text{ and}$$

$$x_i = a + i\Delta x = 0 + \frac{2}{n}i = \frac{2}{n}i.$$

Now we *plug and chug*,

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n f\left(\frac{2}{n}i\right) \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left(\frac{2}{n}i\right)^2 \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \frac{4}{n^2} i^2 \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \cdot \frac{4}{n^2} \left(\sum_{i=1}^n i^2\right) \\ &= \lim_{n \rightarrow \infty} \frac{8}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \lim_{n \rightarrow \infty} \frac{8n(n+1)(2n+1)}{6n^3} \\ &= \lim_{n \rightarrow \infty} \frac{16n^3 + 24n^2 + 8n}{6n^3} \\ &= \frac{8}{3} \end{aligned}$$

□

Exercise: Find the area under the curve $y = 3x$ from $x = 0$ to $x = 4$. Check your answer by drawing the graph and using your knowledge of geometry.

Exercise: Find the area under the curve $y = x^3$ from $x = 1$ to $x = 2$.

Definite integrals: This is basically just a matter of notation. The *definite integral* of the function f from a to b is the area under the curve $y = f(x)$ from $x = a$ to $x = b$. We write

$$\int_a^b f(x) dx = \text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

An integral is like a continuous sum, so you can think of the stretched out S symbol as standing for sum. If you compare the notation to the Riemann sum, you can see how they match up. The S corresponds to the limit and Sigma combined, $f(x)$ to $f(x_i^*)$, and dx to Δx_i .

Example: Based on our previous examples we can write

$$\int_0^2 x^2 dx = \frac{8}{3}$$

to represent the area underneath the curve $y = x^2$ from $x = 0$ to $x = 2$.

Example: Calculate $\int_{-2}^2 \sqrt{4-x^2} dx$.

We need to find the area under the curve $y = \sqrt{4-x^2}$ from $x = -2$ to $x = 2$. If we square both sides of the equation we see that $y^2 = 4 - x^2$, or that $x^2 + y^2 = 4$. This is the equation of a circle centered at the origin with radius 2. Since our original equation was only the positive square root, then its graph is really just the top half of the circle. Therefore the area under the curve is just half of the area of the circle with radius 2. That is,

$$\int_{-2}^2 \sqrt{4-x^2} dx = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi (2)^2 = 2\pi. \quad \square$$

Exercise: Calculate the definite integrals by drawing the graph of each function and using formulas from geometry.

1. $\int_0^1 5x dx$

2. $\int_2^4 x + 3 dx$

3. $\int_{-9}^9 \sqrt{81-x^2} dx$

Definite integrals as antiderivatives: Consider the definite integral

$$\int_a^b f(x) dx,$$

and let F be any antiderivative of f . Remember that this means $F'(x) = f(x)$. Let x_0, x_1, \dots, x_n be any partition of the interval $[a, b]$, so that $a = x_0$ and $b = x_n$. Now consider $F(b) - F(a)$. By adding and subtracting $F(x_i)$ for each partition point x_i , we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) - \dots + F(x_1) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \dots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^n [F(x_i) - F(x_{i-1})] \end{aligned}$$

This is called a *telescoping* sum because the middle terms all squish together and cancel each other out, just like when Jack Sparrow closes his telescope.

Now we need to remember the *Mean Value Theorem* for derivatives. In the notation of this problem, it says that for any differentiable function $F(x)$ on a closed interval $[x_{i-1}, x_i]$, there is a number x_i^* in that interval such that

$$F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}, \text{ or}$$

$$F'(x_i^*)(x_i - x_{i-1}) = [F(x_i) - F(x_{i-1})].$$

But $F'(x_i^*) = f(x_i^*)$ since F is an antiderivative of f , and $(x_i - x_{i-1}) = \Delta x_i$ by our previous considerations. This means that we can write

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^n [F(x_i) - F(x_{i-1})] \\ &= \sum_{i=1}^n f(x_i^*) \Delta x_i \end{aligned}$$

This equation holds for *any* partition, so we can take the limit of each side as n tends to infinity. The left hand side does not have any n 's, so it doesn't change. On the right hand side we get a Riemann sum,

$$F(b) - F(a) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx.$$

So we have just shown that

$$\int_a^b f(x) dx = F(b) - F(a)$$

for any antiderivative F of f . This result is sometimes called the *Evaluation Theorem*. It gives us an easy way to calculate the area under a curve without actually evaluating a Riemann sum; provided of course that we can find an antiderivative.

Example: Let's use this new idea on the same example we've been doing throughout the notes. We know that $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$. Actually, $F(x) = \frac{1}{3}x^3 + C$ for any real number C is an antiderivative of f , but since we can choose *any* antiderivative we may as well choose the one with $C = 0$. Now, the evaluation theorem that we just proved says

$$\int_0^2 x^2 dx = F(2) - F(0) = \frac{1}{3}2^3 - \frac{1}{3}0^3 = \frac{8}{3} - 0 = \frac{8}{3}.$$

That's a little easier than calculating the Riemann sum, isn't it? But we never could have proved the theorem without Riemann sums, so don't bad mouth them. The math gods will hear you. And we don't want to upset the math gods.

Exercise: Use antiderivatives to calculate the definite integrals:

1. $\int_0^2 (4x^3 + 3x^2 - 2x) dx$

2. $\int_1^e \frac{1}{x} dx$

3. $\int_0^{\pi/2} \cos x dx$

4. $\int_0^{\pi/4} \sec^2 \theta d\theta$

A special antiderivative: Let f be a continuous function on the interval $[a, b]$, and define a new function

$$g(x) = \int_a^x f(t) dt$$

for all x in the interval. If F is any antiderivative of f , then the evaluation theorem tells us

$$g(x) = F(x) - F(a).$$

Taking the derivative of both sides with respect to x we obtain

$$g'(x) = \frac{d}{dx}[F(x) - F(a)] = F'(x) - 0 = f(x).$$

So we have shown that $g' = f$; hence g is an antiderivative of f . Since $F(a)$ is a number, then $g(x)$ is really just the antiderivative $F(x)$ plus a constant. So it's not a surprise that it's also an antiderivative. In a way it is the most natural antiderivative because it is obtained by integrating f . Together with the previous result, this means that integration and antiderivatives are really just different manifestations of the same process.

This leads us to one of the most important theorems in all of math, the *Fundamental Theorem of Calculus*.

The Fundamental Theorem of Calculus: Let f be a continuous function on the closed interval $[a, b]$, and F any antiderivative of f . Then,

1. $\frac{d}{dx} \int_a^x f(t) dx = f(x)$, and
2. $\int_a^b \frac{d}{dx} [F(x)] dx = F(b) - F(a)$.

In other words, derivatives and integrals are inverse operations of one another; they cancel each other out. This discovery has been called “one of the greatest achievements of the human mind.” Considering the technological explosion and scientific understanding that calculus has led to, it’s hard to argue with that comment.

We’ve actually proved both parts of this theorem, so you can believe it without having to take anyone else’s word for it. The notation above looks a little different than what we proved, but you can easily check that the results are the same. I prefer this notation because it makes it very clear that the derivative and integral are acting on one another, and canceling each other out. You may also see the theorem written more in the spirit of how we proved it, like this:

Let $f(x)$ be a continuous function on the closed interval $[a, b]$. Then for all x in the interval $[a, b]$, the following are true

1. The function $g(x) = \int_a^x f(t) dt$ is a continuous antiderivative of $f(x)$, and
2. $\int_a^b f(x) dx = F(b) - F(a)$ where F is any antiderivative of f .

Example: Find $\frac{d}{dx} \int_1^{x^2} f(t) dt$.

Using part 1 of the FTC, we can write $g(x) = \int_1^x f(t) dt$. Then $g(x^2) = \int_1^{x^2} f(t) dt$, and

$$\frac{d}{dx} \int_1^{x^2} f(t) dt = \frac{d}{dx} [g(x^2)] = g'(x^2) \cdot 2x = f(x^2) \cdot 2x$$

by using the chain rule.

Using this idea we can make a general chain rule formula for these types of problems,

$$\frac{d}{dx} \int_a^{u(x)} f(t) dt = f(u(x)) \cdot u'(x)$$

Exercise: Show that $\frac{d}{dx} \int_0^{\sin x} \sqrt{1-t^2} dt = \cos^2 x$.

Indefinite integrals: At this point we have already shown that definite integrals are essentially equal to antiderivatives. We've also shown that definite integrals and derivatives are inverse operations of one another, so the name "antiderivative" is completely justified. However, we don't have a good notation for antiderivatives. On the last page alone we used both F and g to denote antiderivatives of f . The Fundamental Theorem of Calculus suggests that we should write

$$\int f(x) dx = F(x) + C$$

where we leave the limits off of the integral sign, and replace $F(a)$ by an arbitrary (unknown) constant C . We call this the *indefinite integral* of f . Notice that $F(x) + C$ represents *all* antiderivatives of f . We always need the "+ C ." Without it our answer is wrong.

Important note: A definite integral is an area, hence a number. An indefinite integral is an antiderivative, hence a function. In other words, even though they are both called *integrals*, they are very different objects and cannot be confused or interchanged.

Exercise: Find the indefinite integrals

1. $\int \cos x dx$

2. $\int \cot \theta \csc \theta d\theta$

3. $\int (3x^2 + 6x - 5) dx$

Final remarks: At the beginning of these notes we required that $y = f(x)$ be a positive, continuous function. This was to make sure that all of our area considerations made sense. By the time we got to the fundamental theorem we had dropped the hypothesis that f be positive. In reality, it's okay if f is occasionally negative. If the graph of f lies below the x -axis, we consider the area between f and the x -axis to be negative. Subtract the negative areas from the positive ones, and everything will work out fine.

Moreover, f doesn't even have to be continuous. It just has to be piece-wise continuous. If the graph has holes or jumps, just calculate the areas on each side of the discontinuities separately, then add them together. However, if f is discontinuous because it has a vertical asymptote, then that is bad. Sometimes we can still calculate the definite integral, and sometimes we cannot. You'll learn how to deal with situations like this in Calculus II.